ON CHARACTERIZATIONS AND STRUCTURE OF INTERVAL DIGRAPHS AND UNIT PROBE INTERVAL GRAPHS

by

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On Characterizations and Structure of Interval Digraphs and Unit Probe Interval Graphs

Thesis directed by Professor J. Richard Lundgren

ABSTRACT

Interval graphs and their variations have been studied extensively for the last 50 years from both a theoretical standpoint and due to their importance in applications. In this thesis we will explore several variations of interval graphs and unit interval graphs including interval digraphs, interval bigraphs and probe interval graphs. Interval graphs were mathematically introduced by Hajos [22]. The origin of interval graphs may also be regarded as an application to the research of Benzer in 1959 [1] in his study of the structure of bacterial genes. Many nice characterizations have been found for interval graphs, but the same cannot be said about the various generalizations. The work presented in this thesis will indicate that characterization of the broadest classes of the generalizations poses a very difficult problem and so the focus in this thesis is on subclasses of interval digraphs, interval bigraphs, and probe interval graphs.

Interval digraphs were introduced by Sen et al. in [39](1989). In 1998 West gave adjacency matrix characterizations of interval digraphs and unit interval digraphs [42]. So far, in the most generic sense, no forbidden subgraph characterization of interval digraphs has been been found, but those tournaments which are interval digraphs have been characterized in various ways by Brown, Busch and Lundgren in 2007. In this thesis we generalize some of their results to other classes of interval digraphs.

A natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [23] in 1982. Unit and proper interval graphs were

introduced and characterized by Roberts in 1969 [36]. He proved that the classes of proper interval graphs and unit interval graphs coincide. Proskurowski and Telle [33] extended the idea of proper interval graphs to p-proper interval graphs. Recently Beyerl and Jamison introduced p-improper interval graphs. They focussed on a special case of p-improper interval graphs. Here we generalize and obtain similar results for p-improper interval bigraphs.

The probe interval graph was invented in order to aid with the task called physical mapping faced in connection with the human genome project, cf. work of Zhang and Zhang et al. [43], [45], [44]. No generalized characterization of probe interval graphs have been found so far. Li Sheng first characterized probe interval graphs for trees[41]. Following this characterization of cycle-free probe interval graphs, in 2009 Brown, Sheng and Lundgren gave a characterization for cycle-free unit probe interval graphs [15]. Recently Brown and Langley characterized unit probe interval graphs for bipartite graphs [10]. In this thesis we give a characterization of 2-trees that are unit probe interval graphs. In 2005 Pržulj and Corneil found at least 62 distinct minimal forbidden induced subgraphs for probe interval graphs that are 2-trees [34]. More recently Brown, Flesch and Lundgren extended the list to 69 and also gave a characterization [7]. In this thesis we follow the idea of Pržulj and Corneil for the next best case which is for 2-trees. The restriction to 2-trees is a reasonable thing to do because of the progression of the Sheng, Brown-Sheng-Lundgren, and Brown-Langely results, and also because of the Corneil-Pržulj and Brown-Flesch-Lundgren results. We first characterize 2-caterpillars and interior 2-caterpillars in terms of forbidden induced subgraphs and show that 2-trees that are unit probe interval graphs have to be interior 2-caterpillars. Then we look at 2-paths that are unit probe interval graphs and characterize them. Using similar ideas we subsequently characterize interior 2-caterpillar which are unit probe interval graphs. Finally we use these results to get a complete characterization of 2-trees which are unit probe interval graphs.

The form and content of this abstract are approved. I recommend its publication.

Approved: J. Richard Lundgren

DEDICATION

Mom and dad and my sweethearts Isha, Sheila, Anoushka and Bubai.

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1. Introduction

1.1 Background

Interval graphs and their variations have been studied extensively for the last 50 years from a theoretical standpoint and due to their importance in applications. In this thesis we will explore several variations of interval graphs namely interval digraphs, interval bigraphs and probe interval graphs. The work on interval bigraphs will focus on p-improper interval bigraph and the work on probe interval graphs will be on unit probe interval graphs.

Let a graph G have vertex set $V(G)$ and edge set $E(G)$. If $x, y \in V(G)$ are adjacent, then we denote $xy \in E(G)$. If G is bipartite, we denote the partitions of the vertex set as $V(G) = \{X \cup Y\}$. A graph is *interval* if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. Interval graphs were theoretically introduced in 1957 by Hajos [22] and also appeared in applied research by Benzer in 1959 [1] in his study of the structure of bacterial genes. Interval graphs were characterized by the absence of induced cycles larger than 3 and asteriodal triples by Lekkerkerker and Boland [26] in 1962. An *asteroidal triple* (AT) in G is a set A of three vertices such that between any two vertices in A there is a path in G that avoids all neighbors of the third.

Other useful characterizations of interval graphs were given by Gilmore and Hoffman in 1964 [19] and Fulkerson and Gross [17]. Extensive study and research has been done on interval graphs for several decades by both mathematicians and computer scientists. These graphs are used to provide numerous models in diverse areas such as genetics, psychology, sociology, archaeology, or scheduling. For more details on interval graphs and their applications, see books by Roberts [36], Golumbic [20] and Mckee and McMorris [29].

Interval digraphs were introduced by Sen et al. in [39](1989). Langley, Lundgren, Merz gave results on Competition Graphs of Interval Digraphs (1995) in [25]. They showed that the competition graph of an interval digraph is an interval graph and that every interval graph is in fact the competition graph of some interval digraph. Lin, Sen and West gave some interesting results on Interval digraphs and (0,1)-matrices in [27]. In 1998 West gave adjacency matrix characterizations of interval digraphs and unit interval digraphs [42]. At this point there is no forbidden subgraph characterization of interval digraphs, but recently, in 2007, interval tournaments were characterized by a complete list of forbidden subtournaments by Brown, Busch, and Lundgren [6]. In chapter 2 we generalize some of the results from this paper.

A natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [23] in 1982. Let G be a bipartite graph with bipartition $X \cup Y$; we may write $G = (X, Y, E)$ to denote this. A bipartite graph G is an interval bigraph if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X$ and $y \in Y$. Interval bigraphs have been studied by several authors ([9], [14], [23], [24], [27], and [31]). Initially it was thought that the natural extension of asteriodal triples of vertices to asteriodal triples of edges along with induced cycles larger than 4 would give a forbidden subgraph characterization [23]. However, Müller [31] found insects and Hell and Huang [24] found edge asteriods and bugs as forbidden subgraphs, and to date a complete characterization remains elusive. Three edges a, c and e of a graph G form an *asteriodal triple of edges* (ATE) if for any two there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge. Cycle free interval bigraphs were characterized by Brown et al in 2001 [14], and ATEs played a significant role in that characterization. In 2002 a generalization of interval bigraph called an interval k-graph was introduced by Brown et al [8]. More recently, in 2010, Lundgren and Tonnsen characterized 2-trees that are interval k-graphs [28].

A graph is a probe interval graph if there is a partition of $V(G)$ into sets P and N and a collection $\{I_v : v \in V(G)\}$ of intervals of R such that, for $u, v \in V(G)$, $uv \in E(G)$ if and only if $I_u \cap I_v \neq \phi$ and at least one of u or v belongs to P. The sets P and N are called the probes and nonprobes, respectively. An interval graph is a probe interval graph with an empty non-probe set and this class of graphs has been studied extensively. The probe interval graph model was invented in order to aid with the task called physical mapping faced in connection with the human genome project (cf. work of Zhang and Zhang et al. [43], [45], [44]). The paper by McMorris, Wang and Zhang [30] has results similar to those for interval graphs found in [17] by Fulkerson and Gross and [20] by Golumbic. For example interval graphs are chordal while probe interval graphs are weakly chordal and maximal cliques are consecutively orderable in interval graphs while quasi-maximal cliques works in probe interval graphs. In 1999 Li Sheng first characterized cycle-free probe interval graphs [41] The classes of graphs related to probe interval graphs are discussed in [11] by Brown and Lundgren, [8] by Brown, Flink and Lundgren, and [21] by Golumbic and Lipshteyn. Relationships between bipartite probe interval graphs, interval bigraphs and the complements of circular arc graphs are presented in [11]. The problem of characterizing generic probe interval graphs in terms of forbidden subgraphs for now appears to be difficult. Since trees were the only class of graphs where probe interval graphs were characterized, a natural next step was to look at a class of 2-trees. In 2005 Pržulj and Corneil attempted a forbidden subgraph characterization of 2-trees that are probe interval graphs and they found at least 62 distinct minimal forbidden induced subgraphs for probe interval graphs that are 2-trees [34]. More recently Brown, Flesch and Lundgren extended the list to 69 and gave a characterization in terms of sparse spiny interior 2-lobsters [7].

In the last 40 years a subclass of interval graphs has been investigated and studied extensively. This class is the class of unit interval graphs introduced by Roberts in 1969. A unit interval graph is an interval graph with all intervals in some interval representation having the same length. A proper interval graph is an interval graph for which there is an interval representation with no interval properly containing another. Roberts [36] proved that the classes of proper interval graphs and unit interval graphs coincide and he showed that interval graphs that contain no $K_{1,3}$ are unit interval graphs. In 1999 Bogart and West gave a shorter proof of the same result of the equality of proper and unit for interval graphs [4]. They gave a constructive proof of this result, where proper intervals are gradually converted into unit intervals. Much more recently in 2007 Gardi gave a much shorter new constructive proof of Roberts original characterization of unit interval graphs [18].

A unit interval bigraph is an interval bigraph with all intervals in some interval representation having the same length and a proper interval bigraph is an interval bigraph for which there is an interval representation with no interval containing another properly. Several characterizations of proper interval bigraphs have been found in the last decade including one by Lin and West [27]. The idea of proper interval graphs was naturally extended to p-proper interval graphs by Proskurowski and Telle [33]. The p-proper interval graphs are graphs which have an interval representation where no interval is properly contained in more than p other intervals. Any 0-improper interval graph is a proper interval graph and it is easy to check that $K_{1,3}$ is a is a 1-improper interval graph. In chapter 3 we will generalize the class of p-improper interval graph to p-improper interval bigraph. Unit interval digraphs were characterized in 1997 by Lin et al [27]. In 2002, Brown et al [9] conjectured a characterization of unit interval bigraphs. This conjecture was proved by Hell and Huang [24] in 2004. More recently in 2011 Brown and Lundgren gave several additional characterization of unit interval bigraphs [12]. Currently Brown, Flesch and Lundgren are working on unit interval k-graphs for 2-trees. Following up Sheng's characterization of cycle-free probe interval graphs, in 2009, Brown, Lundgren and Sheng gave a characterization of cycle-free unit probe interval graphs [15]. Two natural extensions arise, the first is to characterize bipartite graphs that are unit probe interval graphs and secondly to characterize 2-trees that are unit probe interval graphs. Recently Brown and Langley characterized unit probe interval graphs for bipartite graphs [10]. In chapter 4 we give a characterization of 2-trees that are unit probe interval graphs.

1.2 Introduction to the research

We will introduce notation as we move through the thesis. At the initial stage we will use conventional definitions. As mentioned earlier the study of interval graphs has long been a much researched area in Graph Theory. The majority of work in this thesis is an effort to characterize various classes of variations of interval graphs and unit interval graphs. In chapter 2 we study different classes of digraphs and determine the scenarios when they are bound to be interval. Since the problem of finding a complete characterization of interval digraphs is extremely difficult, we concentrate on specific classes of interval digraphs. We focus our research in this chapter on a paper of Brown, Busch, and Lundgren [6] where a complete characterization of interval tournaments is found. We look at oriented graphs with certain properties and find restrictions that need to be imposed on its structure for it to be interval. We find several classes of directed graphs on n vertices which are interval with restrictions such as containing transitive sub-tournaments on $(n-2)$ or $(n-1)$ vertices and specific adjacencies between vertices.

As stated earlier, another natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris in 1982. The p-improper interval graphs, where no interval contains more than p other intervals, were investigated by Beyerl and Jamison [3]. In chapter 3 we extend the idea by introducing p-improper interval bigraphs, where no interval contains more than p other intervals of the same partite set. In this chapter we determine restrictions on the structure of an interval bigraph for it to be a p-improper interval bigraph. We also study special classes of p-improper interval bigraphs.

In chapter 4, we look at unit probe interval graphs. Since they have been characterized for trees and bipartite graphs , we follow the techniques of Pržulj and Corneil for the next best case which is for 2-trees. We characterize 2-trees that have a unit probe interval representation. In order to accomplish this we first characterize 2-caterpillars and interior 2-caterpillars in terms of forbidden induced subgraphs and show that 2-trees that are unit probe interval graphs have to be interior 2-caterpillars. At this point we realized that the problem was much more difficult than we originally anticipated it to be. So we look at 2-paths that are unit probe interval graphs and characterize them. Then we use this characterization to find the characterization of interior 2-caterpillar which are unit probe interval graphs. Finally we use these results to get a complete characterization of 2-trees which are unit probe interval graphs.

2. Interval Digraphs

2.1 Introduction

A directed graph, or digraph is a graph in which each edge is assigned a direction. An arc (directed edge) from vertex u to vertex v will be denoted $u \mapsto v$, and we will say that u beats v. The set of vertices of a digraph D will be denoted $V(D)$, and its size will be denoted $\mid V(D) \mid$. A subdigraph of D is a digraph consisting of a subset of $V(D)$, with only arcs from D between the vertices in this subset. Throughout this paper, we will only be considering digraphs that have no 2- cycles or loops; so for two vertices $u, v \in V(D)$, if $u \mapsto v$, then we cannot have $v \mapsto u$, or $u = v$

A tournament T is an oriented complete graph, so for any $x, y \in V(T)$ either $x \mapsto y$ or $y \mapsto x$, but never both, and $x \mapsto x$ can never happen. An interval tournament is a tournament that is an interval digraph. Naturally a subtournament of some digraph D is a subdigraph of D that happens to be a tournament.

A directed graph D is an interval digraph if for each vertex u there corresponds an ordered pair of intervals (S_u, T_u) such that for any two vertices $u, v \in V(D)$, $u \mapsto v$, if and only if $S_u \cap T_v \neq \phi$ as shown in the Figure 2.1. Interval digraphs were introduced in 1989 by Sen, Das, Roy and West. They introduced interval digraphs as an analogue of Interval Graphs [39] and gave several characterization of interval digraphs in terms of consecutive ones property of certain matrices, adjacency matrix and Ferrers digraphs. In the same year Sen, Das and Roy gave another adjacency matrix characterization of special types of digraphs which are very close to interval digraphs called circular arc digraphs [40]. In circular-arc digraphs the representations are made by arcs of a circle instead of intervals of real line. Also in 1997, Lin, Sen and West gave some strong results on classes of interval digraphs and 0,1-matrices [27]. In 1998 West again gave a much shorter version of the adjacency matrix characterization of interval digraphs [42]. So far there is no forbidden subgraph characterizations of interval digraphs, but recently, in 2007, interval tournaments have been characterized with a complete list of forbidden subtournaments by Brown, Busch, and Lundgren [6]. They show that a tournament on n vertices is an interval digraph if and only if it has a transitive $(n-1)$-subtournament. If a digraph D is not a tournament, then it may be an interval digraph even if it does not contain a transitive $(n-1)$-subtournament as a subdigraph, as long as there are specific restrictions on D.

We explore what restrictions we can place on D to guarantee that it is interval, and in particular we investigate a broader class of oriented graphs on n vertices that contain a transitive $(n-2)$-tournament as a subdigraph. A complete characterization of interval digraphs appears to be really difficult but we managed to find classes of oriented graphs that are interval digraphs. Thus we investigate different types of digraphs (most of which have characteristics in common with tournaments) to try and determine classes of digraphs that can be shown to have an interval representation.

For a graph G that is not a directed graph, an adjacency between vertices $u, v \in V(G)$ will be denoted as $u \leftrightarrow v$. A bipartite graph, or bigraph, B is a graph in which the vertices are partitioned into two sets X and Y, such that $X \cup Y = V(B)$, and any two vertices $u, v \in V(B)$ can only be adjacent if one vertex from u, v is in X and the other is in Y (and this does not guarantee that they will be adjacent). A bipartite graph B is an interval bigraph if to each vertex there corresponds an interval such that two vertices are adjacent if and only if their corresponding intervals intersect and each of these two vertices belongs to a different partite set as shown in Figure 2.3. In 2004, Hell and Huang gave some interesting results on interval bigraphs [24]. More interesting work has been done on interval bigraphs and other related graphs by Brown and Lundgren in 2006 [11]. Methods for recognition of interval bigraphs and interval digraphs have also been investigated in [31] by Müller. He gave a dynamic programming algorithm recognizing interval bigraphs (interval digraphs) in polynomial time. This algorithm recursively constructs a bipartite interval representation of a graph from bipartite interval representations of proper subgraphs. We will see later that the models for interval digraphs and interval bigraphs are basically same. We use the equivalence of the models for interval digraphs and interval bigraphs in our investigation of which oriented graphs are interval digraphs.

An adjacency matrix of a digraph D, denoted $A(D)$, is a 0, 1-matrix that has a 1 in the entry $a_{i,j}$ (row i, column j) if and only if $v_i \mapsto v_j$for the two vertices $v_i, v_j \in V(D)$. A 0, 1-matrix has a zero partition if after independent row and column permutations every zero can be labeled as C or R, such that below every C is another C (except for C's in the bottom row) and to the right of every R is another R (except for R's in the far right column). Figure 2.2 gives a zero partition of a matrix.

We take into account important results on interval digraphs and interval bigraphs and point out a connection between the two to prove the significant results in this chapter.

The following three theorems are helpful tools needed to show certain digraphs are, or are not, interval.

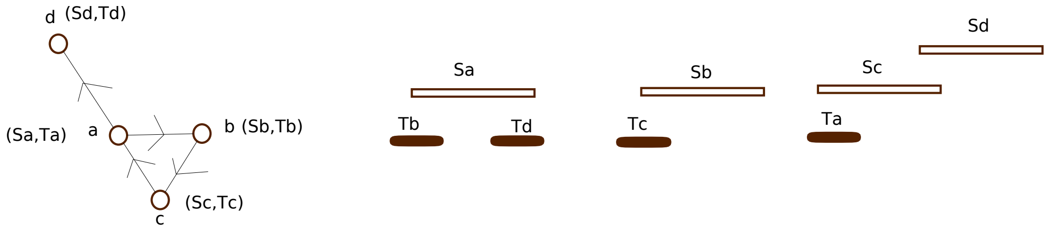


Figure 2.1: Interval digraph

Theorem 2.1.1 *(Sen, Das, Roy, West [39]). A digraph D is an interval digraph if and only if $A(D)$ has a zero partition.*

An asteroidal triple of edges, or ATE, is a set of three edges in a graph for which there is a path between any two of these edges that avoids the neighborhood of the third (the neighborhood of an edge is the set of vertices that are adjacent to one of its end-vertices).

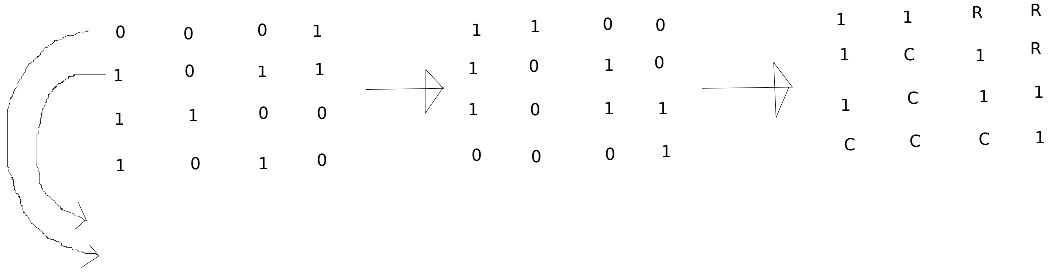


Figure 2.2: Zero partition

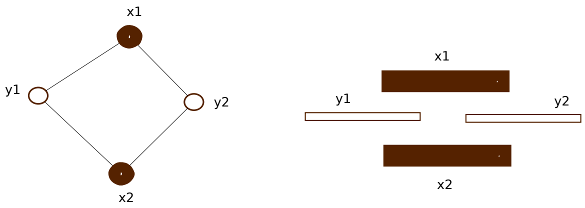


Figure 2.3: Interval bigraph

Theorem 2.1.2 *(Müller [31]). If B is an interval bigraph, then the following hold:*

(a) B has no induced cycle on more than 4 vertices;

(b) B has no asteroidal triple of edges in any induced subgraph;

(c) B has no insect 2.4 as an induced subgraph.

A digraph D can be represented as a bigraph $B(D)$ by letting every vertex $v \in V(D)$ correspond to two vertices in $B(D)$ (one from each partite set) $x_v \in X$ and $y_v \in Y$, such that $u \mapsto v$ in D if and only if $x_u \leftrightarrow y_v$ in B, and this relation accounts for all the edges in $B(D)$. The following theorem puts the concepts of interval digraphs and interval bigraphs together to help us identify digraphs that have no interval representation.

Theorem 2.1.3 *If D is an interval digraph, then $B(D)$ is an interval bigraph.*

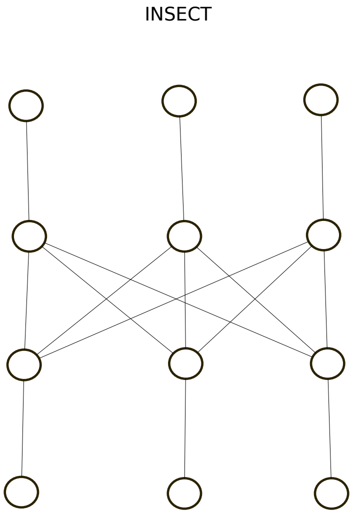


Figure 2.4: Insect

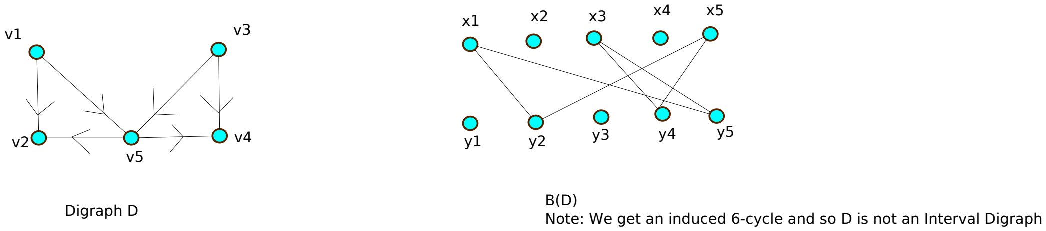

v1
v3
v2
v5
v4
Digraph D
x1
x2
x3
x4
x5
y1
y2
y3
y4
y5
B(D)
Note: We get an induced 6-cycle and so D is not an Interval Digraph

Figure 2.5: Bigraph representation of a digraph

We can get this result by letting $I_{x_v} = S_v$ and $I_{y_v} = T_v$ for any $v \in V(D)$ where x_v and y_v are the corresponding vertices in $B(D)$. So, essentially, the models for interval digraphs and interval bigraphs are the same.

2.2 Motivation from Interval Tournaments

Recently, interval tournaments have been characterized by Brown, Busch, Lundgren [6] and we will use part of a main result proved by them (the following theorem) for our proofs.

Theorem 2.2.1 *(Brown, Busch, Lundgren [6]). Suppose that T is a tournament on n vertices. Then the following are equivalent:*

(a) T is an interval tournament;

(b) T has a transitive $(n-1)$*-subtournament;*

The natural question that rises from this characterization is " What are the other types of digraphs that can be classified as interval?". So we look at digraphs with qualities that are similar to interval tournaments. We know that an interval tournament on n vertices has a transitive $(n-1)$-subtournament. So first we look at digraphs with this same characteristic and this leads us to a generalization of Theorem 2.2.1 $(b) \Rightarrow (a)$ using a similar proof. The following theorem is a generalization of the previous theorem $(b) \Rightarrow (a)$ and we use the concept of zero partition for its proof.

Theorem 2.2.2 *If* D *is a digraph on n vertices with no loops or 2-cycles and a transitive* $(n-1)$*-subtournament, then* D *is an interval digraph.*

Proof:

Let D be a digraph on n vertices with no 2-cycles or loops, and suppose the induced subgraph $D-v$ is a transitive tournament as shown in Figure 2.6. Label the vertices of $D-v$ as $v_{n-1}, v_{n-2}, ..., v_2, v_1$ such that if $v_i \mapsto v_j$, then $i < j$ for $i, j \in 1, 2, ..., n-1$. In forming the adjacency matrix $A(D-v)$, let the rows and columns correspond to this ordering above (from top to bottom and left to right respectively).

Now $A(D-v)$ is lower-triangular with 0's on and above the diagonal and 1's below it. So in every row all the 1's are on the left and all the 0's are to the right of them. Therefore we have a zero-partition for $A(D-v)$ in which all the 0's can be labeled R.

Now attach the v column on the far left and the v row on the bottom of $A(D-v)$. Permute the rows of this new matrix $A(D)$ so that all the 1's in the v column are at the top and all the 0's are below them, which can now be labeled as C's. Note that we do not need to move the v row (at the bottom) to do this since its first entry is

0. Also note that this does not require us to change the R labeling of the 0's from $A(D-v)$ since each of these 0's had a 0 to the right of it, and we have only permuted the rows.

We can now label the 0's in the v row (on the bottom) as C's, since none of them will have 1's below. All of the 0's in $A(D)$ have now been labeled as R or C, so we have a zero-partition, and Theorem 2.1.1 implies that D is an interval digraph.

■

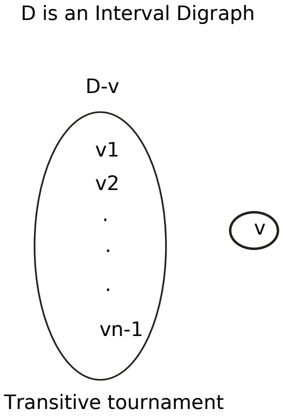


Figure 2.6: $D-v$

Note that if a digraph D on n vertices has a $(n-1)$-subdigraph that is transitive, it is not necessarily an interval digraph. We only know that D has an interval representation in this case if the transitive $(n-1)$-subdigraph is actually a tournament. In fact, a transitive digraph itself may not be interval. An example of this is in Figure 2.8, which shows a transitive digraph D that is not interval (by Theorem 2.1.2) because its bigraph $B(D)$ has an induced 6-cycle.

Suppose now for a digraph D, the induced subgraph $D-v$ is the union of k disjoint transitive tournaments for some $k \in N'$. We find that there do exist several such digraphs that are not interval digraphs. We found 6-cycles, 8-cycles, 10-cycles and even ATEs in the corresponding bigraphs of such digraphs.

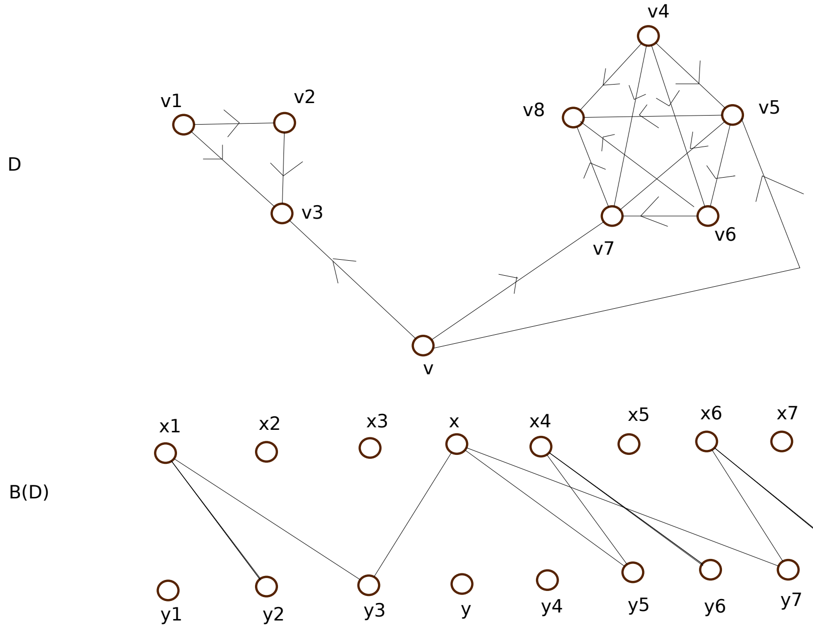
In Figure 2.9, the induced 10-cycle in the bigraph $B(D)$ shows that D is not an interval digraph. However, it has been showed that the 10-cycle is the largest induced cycle that can exist in the bigraph $B(D)$ of such a digraph D [16]. Thus 6, 8, and 10-cycles are the only cycles in the bigraph $B(D)$ that would produce forbidden structures. It is worth mentioning here that it has also been proved that the longest induced path in the bigraph $B(T)$ of any transitive tournament T consists of three edges (and four vertices)[16].

Unfortunately, we also discovered that there are many ways that ATE's can appear in the bigraphs corresponding to digraphs D in which $D-v$ is a union of disjoint transitive tournaments. One example is shown below in Figure 2.7. We know that that if $B(D)$ has an ATE, then $B(D)$ is not an interval bigraph, and hence D is not an interval digraph. These results lead us to consider a different generalization, namely, digraphs D on n vertices such that the induced subdigraph on (n-2) vertices given by D-$\{v^{'}, v^{''}\}$ is a transitive tournament.

2.3 Digraphs with a Transitive $(n-2)$- subtournament

A digraph D on n vertices with a transitive $(n-2)$-subtournament can have many structures in its bigraph $B(D)$ that prevent it from being an interval digraph. Figure 2.10 below is an example where we get a forbidden subgraph in $B(D)$ (which is a 6-cycle) for which D-$\{v^{'}, v^{''}\}$ is a transitive tournament. It is also worth-mentioning here that for this particular digraph D, adding any combination of the dashed arcs (or none of them) will result in the same induced 6-cycle in $B(D)$. Altogether, there are a total of 36 possible combinations of these dashed edges that will result in this particular induced 6-cycle.

The example in the Figure 2.10 shows that we can easily end up with an induced 6- cycle in $B(D)$ for a digraph D in which D-$\{v^{'}, v^{''}\}$ is a transitive tournament. Furthermore this type of digraph D has many forbidden structures that will prevent



induced ATE
The darker edges x1y2, x4y6, x6y8 are the 3 edges

Figure 2.7: ATE in the bigraph

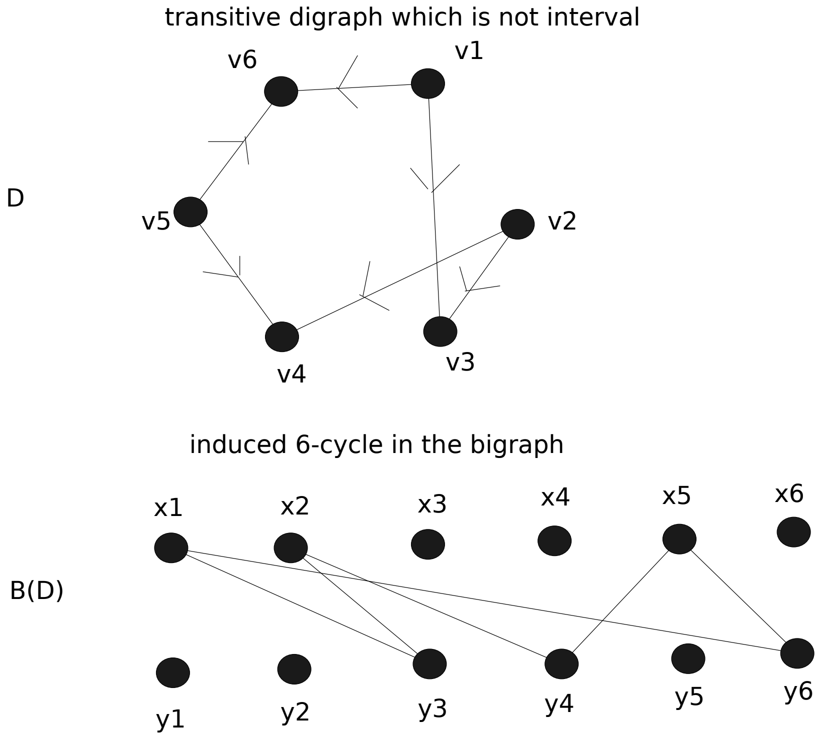
it from having an interval representation (including many induced cycles of length at least 6 and many ATE's in its bigraph). This suggests that perhaps we need more restrictions on D to find a type of digraph that we know is interval. The following theorem places more restrictions on the arcs directed between the vertices of the transitive $(n-2)$-subtournament and the other two vertices $v^{'}$ and $v^{''}$.

Theorem 2.3.1 *Let D be a directed graph on n vertices with no 2-cycles or loops, and with a transitive $(n-2)$-subtournament D^{*}=D-$\{v^{'}, v^{''}\}$. Suppose either (a) or (b) is true, but not both:*

(a) Suppose no vertices in D^{*} beat $v^{'}$ or $v^{''}$, and $v^{''}$ beats a subset of the vertices in D^{*} that $v^{'}$ beats;

(b) Suppose no vertices in D^{*} are beaten by $v^{'}$ or $v^{''}$, and $v^{''}$ is beaten by a subset of the vertices in D^{*} that beat $v^{'}$.

Then D is an interval digraph


transitive digraph which is not interval

induced 6-cycle in the bigraph

Figure 2.8: 6-cycle

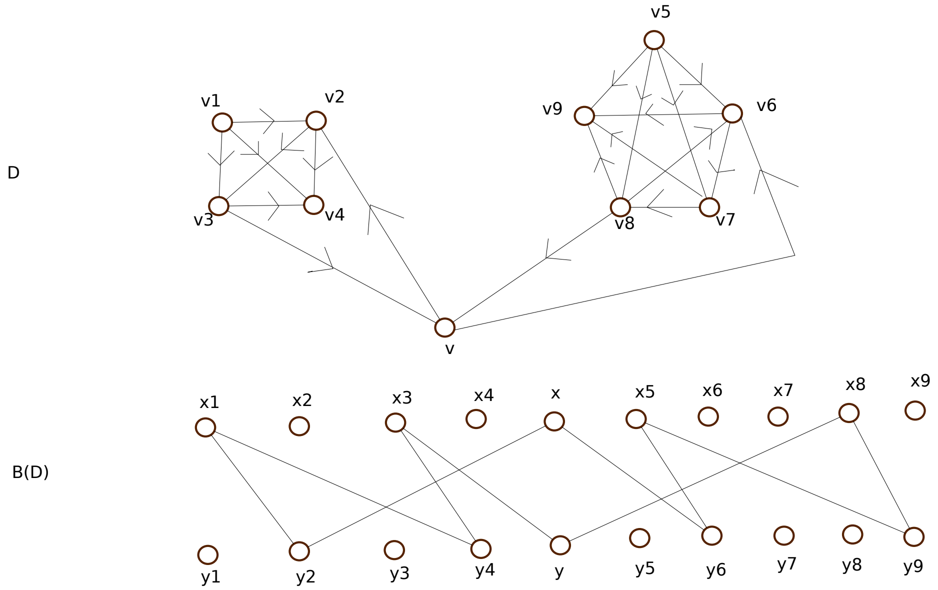
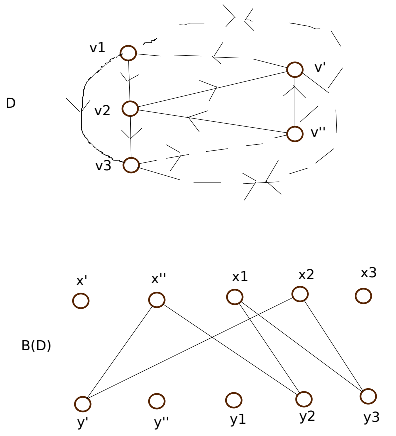


Figure 2.9: 10-cycle



Any combination of the broken arcs also forms a 6 cylcle in B(D)
In total we can get 36 ways of forming an induced 6-cycle in B(D)

Figure 2.10: Induced 6-cycle in the bigraph representation

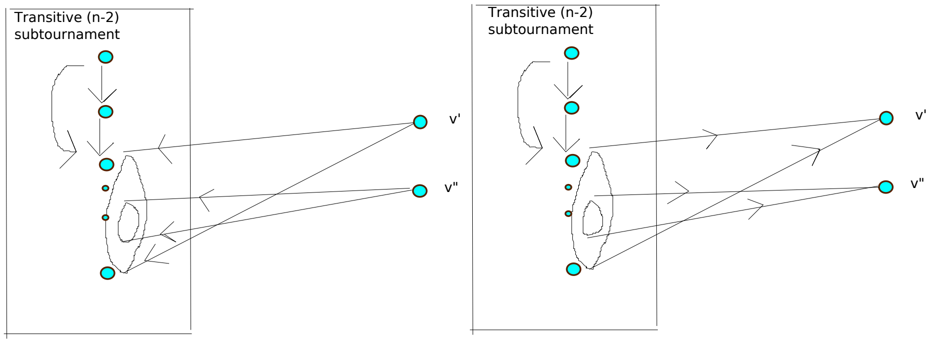


Figure 2.11: More restrictions on the arcs

Proof: Let D be a digraph on n vertices with no 2-cycles or loops, and suppose D^*=D-$\{v', v''\}$ is a transitive tournament. Thus the subdigraph D-v'' has a transitive $(n-2)$-subtournament, and by Theorem 2.2.1 it is an interval digraph. Theorem 2.1.1 now implies that $A(D-v'')$ has a zero partition. Assume that D^* meets the conditions of either (a) or (b) above, but not both. The following cases must be considered:

(i) v', and v'' have no arc between them,

(ii)v'' has an arc directed to v',

(iii) v' has an arc directed to v''.

Arrange the rows and columns of $A(D - v^{''})$ as in the proof of Theorem 2.2.1 Now add the column and row corresponding to $v^{''}$ at the far left (with column $v^{'}$ to its right) and bottom of the matrix (with row $v^{'}$ above), respectively. Recall from the proof of Theorem 2.2.1 that the submatrix $A(D^*)$ already has each of its 0s labeled as R.

If (a) is true (but (b) is not):

(i) Let $v^{'}$, and $v^{''}$ have no arc directed between them. Row $v^{'}$ (just above row $v^{''}$) has its 0s labeled as Cs, as in the proof of Theorem 2.2.1. Row $v^{''}$ will have a 0 in every column entry that row $v^{'}$ has a 0, and will also have 0s in some column entries in which row $v^{'}$ has 1s (and 1s in the other entries). So each 0 in row $v^{''}$ can be labeled as C, which will each be below another C or a 1. Also, columns $v^{'}$ and $v^{''}$ will have all 0 entries since no vertices in D^* beat these two vertices, and each of these 0's can be labeled as C. Now $A(D)$ has a zero partition.

(ii) If $v^{''}$ has a arc directed to $v^{'}$, then $a_{v^{''}v^{'}}$ is the only entry that is different from $A(D)$ in (i) above, and it now becomes a 1. This 1 is a problem because it is below the 0s in column $v^{'}$, which had previously been labeled as Cs. So now move column $v^{'}$ to the far right. Because no vertices from D^* have directed arcs to $v^{'}$ or $v^{''}$, the 1 at the bottom of column $v^{'}$ is the only 1 in that column, and all the 0s above it can be relabeled as Rs. Now $A(D)$ has a zero partition.

(iii) If $v^{'}$ has a directed arc to $v^{''}$, then $a_{v^{'}v^{''}}$ is the only entry that is different from $A(D)$ in (i) above, and it now becomes a 1. In the same manner as part (ii), move column $v^{''}$ to the far right and relabel all entries above entry $a_{v^{'}v^{''}}$ (which are all 0s) as Rs. There is also one 0 below $a_{v^{'}v^{''}}$ in column $v^{''}$, which can be labeled as C. Now $A(D)$ has a zero partition.

If (b) is true (but (a) is not):

(i) Let $v^{'}$ and $v^{''}$ have no arc directed between them. The rows of $A(D - v^{''})$ have already been permuted (as in the proof of Theorem 2.2.1) so that the rows with a 1 in the column $v^{'}$ (just right of column $v^{''}$) are at the top, with all the 0's in this column below the 1's and labeled as C's. Now permute these rows with 1's in column $v^{'}$ so that the rows with 1's in column $v^{''}$ are at the top with 0's underneath (note that column $v^{'}$ will still have all its 1's above its 0's). Now all the 0's are below the 1's in column $v^{'}$ and $v^{''}$, and each of these 0's can be labeled as C. The rows $v^{'}$ and $v^{''}$ will be composed entirely of 0's, which can be labeled as C's, since no vertices in D^* are beaten by these two vertices. Now $A(D)$ has a zero partition.

(ii) If $v^{''}$ has a directed arc to $v^{'}$, then $a_{v^{''}v^{'}}$ is the only entry that is different from $A(D)$ in (i) above, and it now becomes a 1. This 1 is now a problem because it is below the 0's in column $v^{'}$, which had previously been labeled as C's. So now move row $v^{''}$ just below the lowest row in $A(D - v^{''})$ for which there is a 1 in column $v^{'}$. Because $v^{'}$ and $v^{''}$ do not have directed arcs to vertices in D^*, $a_{v^{''}v^{'}}$ is the only entry in row $v^{''}$ that is a 1, so the 0 to the left (in column $v^{''}$) is still labeled as a C, and all the 0's to the right can be labeled as R's. Now $A(D)$ has a zero partition.

(iii) If $v^{'}$ has a directed arc to $v^{''}$, then $a_{v^{'}v^{''}}$ is the only entry that is different from $A(D)$ in (i) above, and it now becomes a 1. This 1 is now a problem because it is below the 0's in column $v^{''}$, which had previously been labeled as C's. Now move row $v^{'}$ to the top. Because $v^{'}$ and $v^{''}$ do not have directed arcs to vertices in D^*, $a_{v^{'}v^{''}}$ is the only entry in row $v^{'}$ that is a 1, so all the other entries are 0's, which are all to the right of entry $a_{v^{'}v^{''}}$ and can each be labeled R. Now we have a zero partition for $A(D)$.

In every case above we have found a zero partition of $A(D)$, so Theorem 2.1.1 implies that D is an interval digraph.

■

It is worth mentioning here that if there is no arc directed between $v^{'}$ and $v^{''}$ then D can be proved to be an interval digraph with fewer restrictions placed on the directed arcs between the vertices of D^* and $v^{'}$, $v^{''}$. The following theorem proves this fact.

Theorem 2.3.2 *Let D be a directed graph on n vertices with no 2-cycles or loops, and with a transitive* $(n-2)$*-subtournament* $D^* = D\text{-}\{v^{'}, v^{''}\}$*, where* $v^{'}$ *and* $v^{''}$ *are vertices that have no arc directed between them in D.*
Suppose $v^{'}$ *beats a subset of the vertices in* D^* *that* $v^{''}$ *beats, or* $v^{''}$ *beats a subset of the vertices in* D^* *that* $v^{'}$ *beats. Also suppose that* $v^{'}$ *is beaten by a subset of the vertices in* D^* *that beat* $v^{''}$*, or* $v^{''}$ *is beaten by a subset of the vertices in* D^* *that beat* $v^{'}$*.*
Then D is an interval digraph.

Proof:

Because D is a digraph with no 2-cycles or loops that has a transitive $(n-2)$-subtournament D^*=D-$\{v^{'}, v^{''}\}$, we know that the subgraph D-$v^{''}$ has a transitive subtournament on all but one of its vertices. Theorem 2.2.1 implies that D-$v^{''}$ is an interval digraph, which is equivalent to $A(D-v^{''})$ having a zero-partition from Theorem 2.1.1.

Arrange the submatrix $A(D-v^{''})$ as in the proof of Theorem 2.2.1, where the rows corresponding to vertices from D^* with arcs directed to $v^{'}$ are at the top, so that all the 0's in column $v^{'}$ are below the 1's and each is labeled as C.

If the vertices in D^* with directed arcs to $v^{''}$ are a subset of those with directed arcs to $v^{'}$, then we can place column $v^{''}$ to the left of column $v^{'}$ in A(D) and rearrange the top rows with 1's in column $v^{'}$ so that all the rows with 1's in column $v^{''}$ are on top, with the 0's underneath them labeled as C's. (Note that column $v^{'}$ will still have all its 1's at the top, with 0's underneath and labeled as C's).

If the vertices in D^* with arcs directed to $v^{'}$ are a subset of those with arcs directed to $v^{''}$, then we can place column $v^{''}$ to the left of column $v^{'}$ again. All the rows with 1s in column $v^{'}$ are already at the top, which also have 1's in column $v^{''}$, and we can place the other rows with 1's in column $v^{''}$ directly below these rows so that columns $v^{'}$ and $v^{''}$ have the 0's labeled as C's and placed below all of the 1's.

Since we have arranged $A(D - v^{''})$ as in the proof of Theorem 2.2.1, row $v^{'}$ is at the bottom of $A(D - v^{''})$ and has each of its 0's labeled as C.

If the vertices in D^* with arcs directed from $v^{''}$ are a subset of those with directed arcs from $v^{'}$, then we can place row $v^{''}$ below row $v^{'}$ in $A(D)$ and label each of its 0's as a C, since each of these C's will either be below a 1 or another C.

If the vertices in D^* with arcs directed from $v^{'}$ are a subset of those with arcs directed from $v^{''}$, then we can place row $v^{''}$ just above row $v^{'}$ in $A(D)$ and label each of its 0's as a C, since each of these C's will always be above another C.

As in the proof of Theorem 2.2.1, the rest of the 0's (which are from the submatrix $A(D^*)$ are labeled as R's. Therefore $A(D)$ has a zero partition, and from Theorem 2.1.1 this implies that D is an interval digraph.

■

Theorem 2.3.2 will fail to hold if there exist an arc between $v^{'}$ and $v^{''}$. The figure 2.14 shows an induced 6-cycle in $B(D)$ corresponding to the digraph D which has arc $v^{'} \mapsto v^{''}$. In D, $v^{''}$ beats a subset of the vertices in D-[$v^{'}$,$v^{''}$] that $v^{'}$ beats and $v^{''}$ is beaten by a subset of the vertices in D-[$v^{'}$,$v^{''}$] that beat $v^{'}$.

2.4 More than one nontrivial strong component

A strong component of a digraph D is a maximal strongly connected subgraph. A directed graph is strongly connected if for every pair of vertices v, u there exists at least one path from v to u, and at least one path from u to v. Brown, Busch, Lundgren [6] proved the following theorem about strong components in tournaments.

Theorem 2.4.1 *No tournament that has two or more non-trivial strong components is an interval tournament.*

However, an interval digraph can have more than one nontrivial strong component. In fact, certain digraphs with two or more nontrivial strong components that have specific types of adjacency matrices always have an interval representation, as the following theorem shows. Figure 2.15 shows an interval digraph with 2 non-trivial strong components. We will require a couple of new definitions to prove that certain digraphs with specific adjacency matrices and two or more non-trivial strong components will always be interval.

Definition 2.4.2 *An R zero partition of a {0, 1 }- matrix is a zero partition in which each of the 0's of the matrix can be labeled as R (where to the right of each R is another R) after column permutations. Likewise, a C zero partition is a zero partition in which each of the 0's can be labeled as C (where below each C is another C) after row permutations.*

The following theorem shows that a digraph with exactly two non-trivial strong components will be interval if the adjacency matrices of the strong components and the adjacency matrix determining the relation between the strong components are specific. It is worth mentioning here that the following result has been generalized to a digraph with $k \in N$ non-trivial strong components in [16].

Theorem 2.4.3 *Let D be a directed graph on n vertices with exactly two nontrivial strong components, D_1 and D_2. Suppose that a subset of the vertices of D_1 have arcs directed to a subset of the vertices of D_2 (but no vertices in D_2 have arcs directed to vertices in D_1.*

If the submatrices $A_1 = A(D_1)$ and $A_2 = A(D_2)$ both have a C zero partition, and the submatrix B (whose rows represent the vertices of D_1, and whose columns represent the vertices of D_2) has an R zero partition, then D is an interval digraph.

Proof:

Let the adjacency matrix $A(D)$ be arranged as follows:

$$A(D) =$$

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$

Our hypothesis states that not only does D contain exactly two nontrivial strong components D_1 and D_2, but their adjacency matrices, A_1 and A_2, must have a C zero partition after row permutations. Also, the arcs directed from vertices in D_1 to vertices in D_2 must be such that the adjacency submatrix B (whose rows represent the vertices of D_1, and whose columns represent the vertices of D_2) must have an R zero partition after column permutations. Assuming that these conditions are met, the column permutations used to get an R zero partition of B (which do not affect the order of the entries in the columns of A_1 or A_2) will not change the C labelings of the 0s in A_1 and A_2, and the row permutations used to get a C zero partition of A_1 and A_2 (which do not affect the order of the entries in the rows of B) will not change the R labelings of the 0s in B. Therefore $A(D)$ has a zero partition in which each of the 0s in the submatrices represented by A_1, A_2, and 0 are labeled as C, and the submatrix B has each of its 0s labeled as R. This implies (by Theorem 2.1.1) that D is an interval digraph.

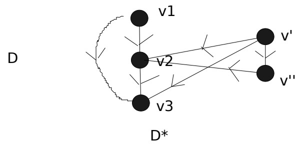
■

We have studied the structures of several digraphs in order to ascertain when they are interval. We focussed on digraphs that possess some similar qualities of interval tournaments such as the digraphs with a transitive subtournament on all but two vertices and the digraphs for which the removal of one vertex leaves a subdigraph that is a transitive sub-tournament. These types of digraphs seemed to be the natural place to start because there is a complete characterization of interval tournaments. We successfully found different restrictions that force a digraph to be interval thus finding some special classes of interval digraphs.

There are still numerous classes of digraphs that will be interval with some restrictions on their arcs which have not been explored yet. Surely the problem of finding a complete forbidden subdigraph characterization of interval digraphs will be a very difficult one since the list of forbidden structures is already very long. Much work still needs to be done on this topic.

No vertices from D* beat v' or v'' and v'' beats a subset of vertices that v' beats

v' has a directed arc to v''



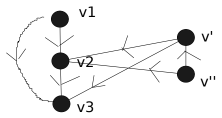
	v''	v'	v3	v2	v1
v3	0	0	0	0	0
v2	0	0	1	0	0
v1	0	0	1	1	0
v'	1	0	1	1	0
v''	0	0	0	1	0

	v'	v3	v2	v1	v''
v3	0	0	0	0	0
v2	0	1	0	0	0
v1	0	1	1	0	0
v'	0	1	1	0	1
v''	0	0	1	0	0

ZERO-PARTITION

	v'	v3	v2	v1	v''
v3	C	R	R	R	R
v2	C	1	R	R	R
v1	C	1	1	R	R
v'	C	1	1	C	1
v''	C	C	1	C	C

v'' has a directed arc to v'



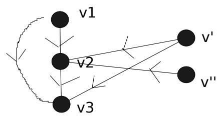
	v''	v'	v3	v2	v1
v3	0	0	0	0	0
v2	0	0	1	0	0
v1	0	0	1	1	0
v'	0	0	1	1	0
v''	0	1	0	1	0

	v''	v3	v2	v1	v'
v3	0	0	0	0	0
v2	0	1	0	0	0
v1	0	1	1	0	0
v'	0	1	1	0	0
v''	0	0	1	0	1

ZERO-PARTITION

	v'	v3	v2	v1	v''
v3	C	R	R	R	R
v2	C	1	R	R	R
v1	C	1	1	R	R
v'	C	1	1	C	R
v''	C	C	1	C	C

No directed arc between v'' and v'



	v''	v'	v3	v2	v1
v3	0	0	0	0	0
v2	0	0	1	0	0
v1	0	0	1	1	0
v'	0	0	1	1	0
v''	0	0	0	1	0

ZERO-PARTITION

	v''	v'	v3	v2	v1
v3	C	C	R	R	R
v2	C	C	1	R	R
v1	C	C	1	1	R
v'	C	C	1	1	C
v''	C	C	C	1	C

Figure 2.12: Example of a digraph satisfying a condition of the theorem 2.3.1 and its zero partition

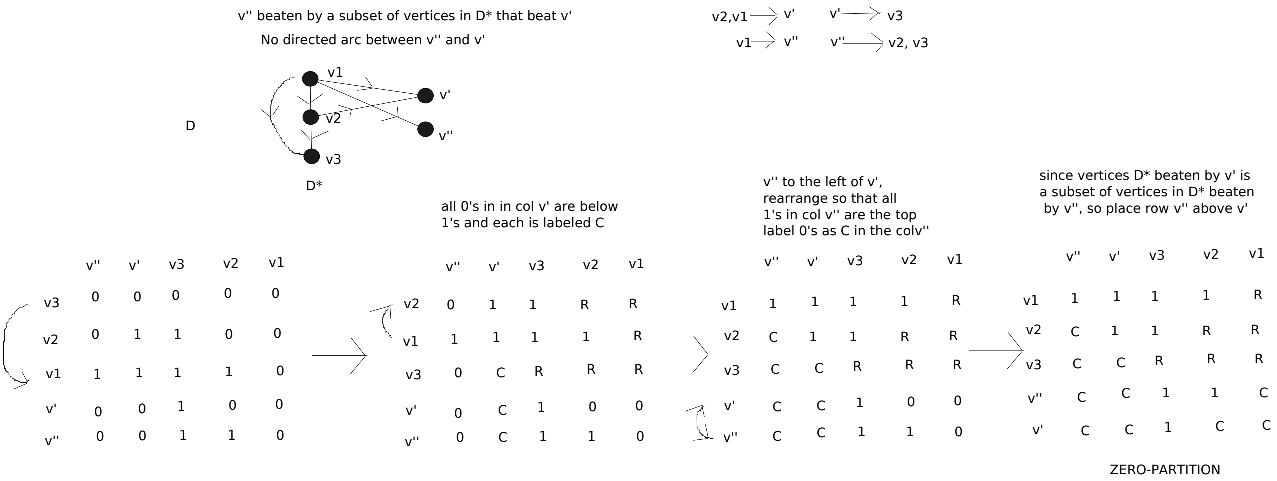


Figure 2.13: Example of a digraph satisfying a condition of the theorem 2.3.2 and its zero partition

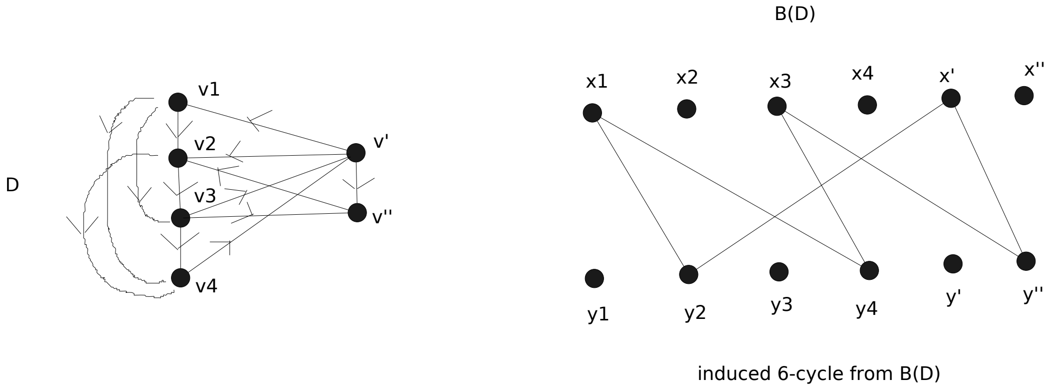


Figure 2.14: Example of a digraph that fails the hypothesis of theorem 2.3.2 since $v^{'}$ beats $v^{''}$ and hence forms a 6-cycle in its bigraph representation

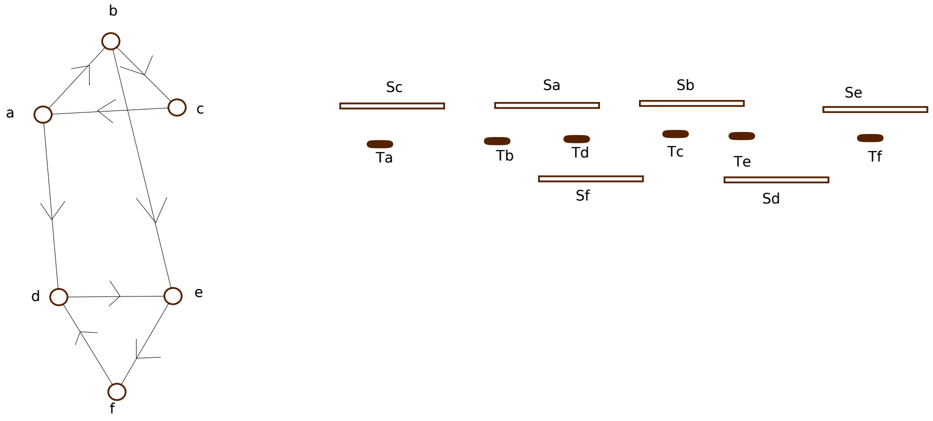


Figure 2.15: Strong components

3. Interval Bigraph Impropriety

3.1 Introduction

An interval graph is proper if and only if it has a representation in which no interval contains another. Beyerl and Jamison introduced the study of p-improper interval graphs where no interval contains more than p other intervals in 2008. Thus a proper interval graph is a 0-improper interval graph. In this chapter we extend the idea by introducing p-improper interval bigraphs, where no interval contains more than p other intervals of the same partite set. Several authors have studied proper interval bigraphs. One of these characterizations has three forbidden subgraphs. We find bounds on the structure of p-improper interval bigraphs and characterize a special case of p-improper interval bigraphs.

Let a graph G have vertex set V and edge set E. $x, y \in V$ being adjacent is denoted by $xy \in E$. A finite simple graph $G(V, E)$ is an interval graph if we can find a mapping $\theta : v \longrightarrow I_v$ from vertices of G to intervals on the real line, such that the edge xy exists if and only if $I_x \cap I_y \neq \phi$ for all $x, y \in V(G)$. Interval graphs were first discussed by Hajos [22]. The study of interval graphs also has its origin in a paper of Benzer (1959) [1] who was studying the structure of bacterial genes.

A well-known characterization of interval graphs was given by Lekkerkerker and Boland in 1962. The Lekkerkerker-Boland Theorem [26] says that chordless cycles and asteroidal triples form a defining class of forbidden subgraphs for the class of interval graphs. An *asteroidal triple* (AT) in G is a set A of three vertices such that between any two vertices in A there is a path between them that avoids all neighbors of the third. A natural extension of AT is an Asteroidal Triple of Edges. An *asteroidal triple of edges* (ATE) is a set of three edges such that for any two there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge. A natural extension of intervals graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [23] in 1982. A bipartite graph

$G(X, Y, E)$ is an interval bigraph if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X$ and $y \in Y$. These graphs have been studied by several authors [13],[27]. To date no forbidden subgraph characterization of interval bigraphs has been found, but initially it was thought that asteriodal triples of edges along with induced cycles larger than 4 would work [23]. They proved that if B is an interval bigraph then B does not have ATE. However, Müller [31] found insects and Hell and Huang [24] found edge asteriods and bugs as forbidden subgraphs, and to date a complete characterization is still not available.

Fred Roberts [35] in 1969 characterized proper interval graphs. Proper interval graphs are graphs which have an interval representation such that no interval contains another. An interval graph is proper if and only if it does not contain $K_{1,3}$ as an induced subgraph. Unit interval graphs are the graphs having an interval representation in which all the intervals have the same length. Claw-free interval graphs are the interval graphs without an induced copy of the claw, $K_{1,3}$. Roberts has also shown that the class of proper interval graphs coincide with the classes of unit interval graphs and claw-free interval graphs.

Proper interval bigraphs are bigraphs which have an interval representations where no interval contains another. Several charactizations of proper interval bigraphs have been found in the last decade. The graphs in Figure 3.4 are the forbidden subgraphs for proper interval bigraphs found by Lin and West [27].

The idea of proper interval graphs was naturally extended to p-proper interval graphs by Proskurowski and Telle [33]. The p-proper interval graphs are graphs which have an interval representation where no interval is properly contained in more than p other intervals. Beyerl and Jamison [3] investigated a variation in containment and introduced p-improper interval graphs where no interval contains more than p other intervals. A p-improper representation of a graph is an interval representation of the graph where no interval contains more than p other intervals. Figure 3.2 is an illustration of a 1-improper interval graph. Thus any 0-improper interval graph is a proper interval graph. It can be easily checked in Figure 3.1 that $K_{1,3}$ is a 1-improper interval graph.

In this chapter we generalize the class of p-improper interval graphs to p-improper interval bigraphs. We look at ideas which Beyerl and Jamison investigated for interval graphs and apply them in interval bigraphs. Some definitions had to be modified since we are working with two partite sets.

A p-improper interval bigraph is an interval bigraph where no interval contains more than p intervals from the same partite set. In order to be able to generalize the ideas for p-improper interval graphs to p-improper interval bigraphs, we restrict the containment of intervals to an individual partite set as shown in the Figure 3.3. Beyerl and Jamison investigated ideas such as impropriety, weight, p-critical and balance for interval graphs, and we investigate the same ideas with some modified definitions for interval bigraphs. In this chapter we find restrictions in the stucture of an interval bigraph for it to be a p-improper interval bigraph. We also study special classes of p-improper interval bigraphs.

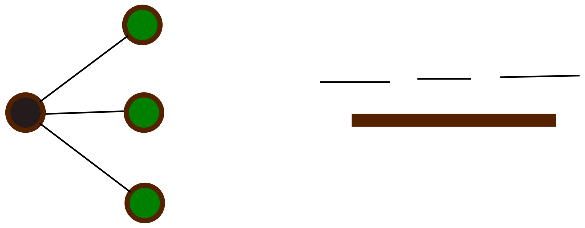


Figure 3.1: $K_{1,3}$

3.2 Impropriety and Weight of Interval Bigraphs

From this point on $B = B(X, Y, E)$ will denote a finite, connected, interval bigraph with bipartition $\{X, Y\}$ and the sets X and Y will be referred to as the partite

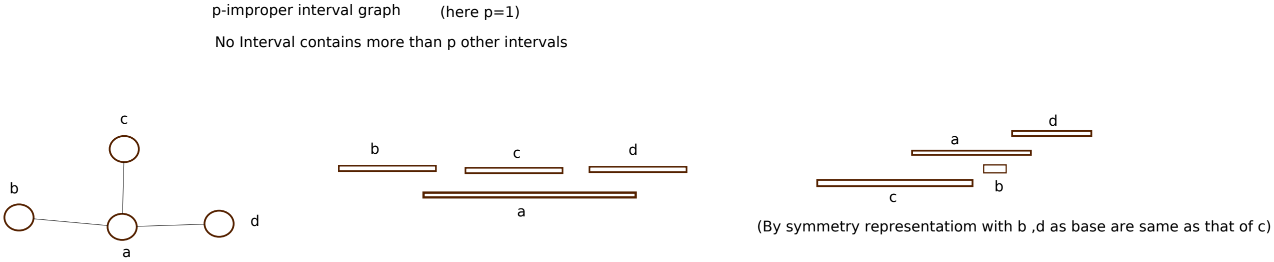

p-improper interval graph
(here p=1)
No Interval contains more than p other intervals
c
b
a
d
b
c
d
a
d
a
b
c
(By symmetry representatiom with b ,d as base are same as that of c)

Figure 3.2: *p*-Improper Interval Graph

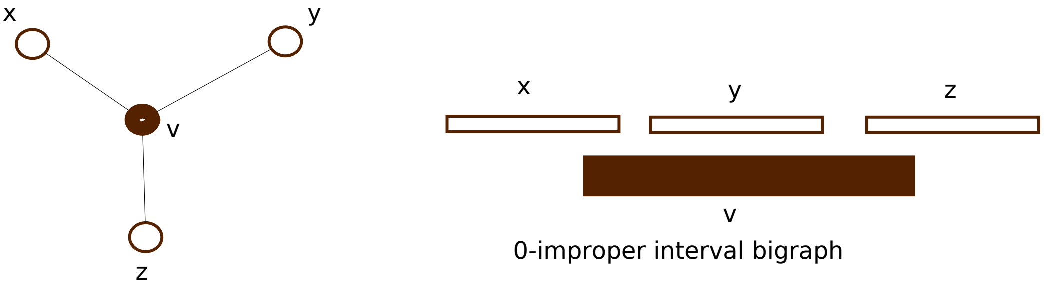


Figure 3.3: 0-Improper Interval Bigraph

sets. A 0-improper interval bigraph is a proper interval bigraph. The following proposition justifies our way of defining a *p*-improper interval bigraph. It explains the sufficiency of restricting the inclusion of intervals to a single partite set in an interval bigraph.

Proposition 3.2.1 *B is a proper interval bigraph if and only if it has an interval representation where no interval from a partite set contains another interval from the same partite set.*

Proof: Suppose $B(X, Y, E)$ is a proper interval bigraph. It follows from the definition that it has an interval representation where no interval from a partite set contains another interval from the same partite set, since no interval contains another.

Suppose B has an interval representation where no interval from a partite set contains another interval from the same partite set. If B is not proper then it has one of the graphs in Figure 3.4 as an induced subgraph [27]. It is easy to check from the interval representation of B1, B2 and B3 (illustrated in Figure 3.4 and Figure 3.5) that they all are forced to have an interval which contains another interval from the same partite set. In Figure 3.4 we avoid the containment of an interval from Y in another interval from Y. This forces the containment of an interval from X in another interval from X. In Figure 3.5 we avoid the containment of an interval from X in another interval from X. This forces the containment of an interval from Y or X in another interval from Y or X. B1 is the only example where we never attain a containment from Y. It always has an interval from X containing another from X. Thus we cannot avoid an X interval properly containing another from X or a Y interval properly containing another from Y. Hence B has an interval which contains another interval from the same partite set which contradicts our assumption. ■

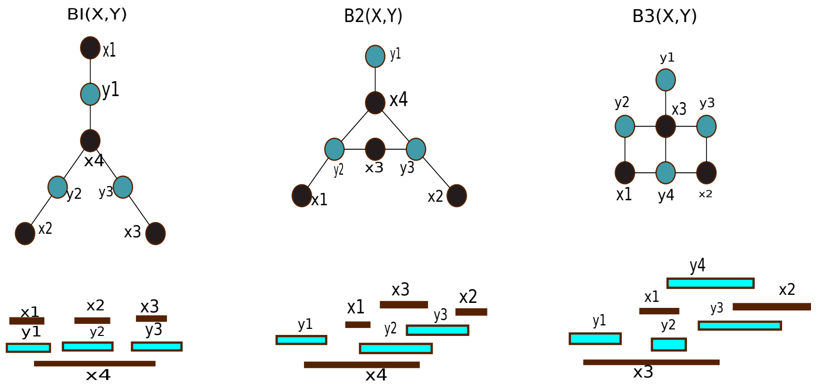


Figure 3.4: Forbidden subgraphs for proper Interval bigraph with their interval representations to force containment from X

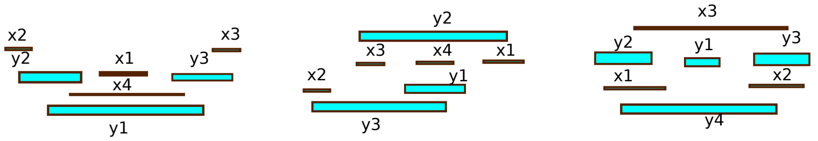


Figure 3.5: Interval representations of B1, B2, B3 to force containment from Y

First we will generalize the notion of impropriety introduced by Beyerl and Jamison. The idea of impropriety comes from p-improper interval bigraphs.

Definition 3.2.2 *The* impropriety of a vertex $z \in X$ *with respect to the interval representation* α *is the number of vertices* $x \in X$ *such that* $I_x \subsetneq I_z$*, which we will denote as* $imp_\alpha(z)$*.*

Definition 3.2.3 *The* impropriety of a representation α *is the maximum* $imp_\alpha(z)$ *for all* $z \in X, Y$ *which we will denote as* $imp_B(\alpha)$*.*

Definition 3.2.4 *The* impropriety of the interval bigraph B *is the minimum* $imp_B(\alpha)$ *over all possible interval representations which we will denote as* $imp(B)$*. A representation which gives the minimum impropriety will be called a minimal representation.*

Any interval bigraph can have an infinite number of interval representations. So it is easy to see that the impropriety is the least p for which the graph is a p-improper interval bigraph as shown in the Figure 3.6. Figures 3.8 and 3.9 give examples of graphs whose impropriety is 1 and 2 respectively. It is easy to check that $I_z \supset I_x, I_y$ is forced.

Definition 3.2.5 *For* $z \in B(X, Y, E)$*, a component of* $B - z$ *will be called a* local component *of* z*. A local component of* z *is* exterior *if and only if it contains an edge* xy *such that* $x, y \notin N(z)$*. (See Figure 3.7).*

It is worth mentioning here that all the bigraphs here have connectivity one.

The following lemma gives us a bound on the number of exterior local components any vertex of an interval bigraph can have.

Lemma 3.2.6 *A vertex* z *in an interval bigraph can have at most two exterior local components.*

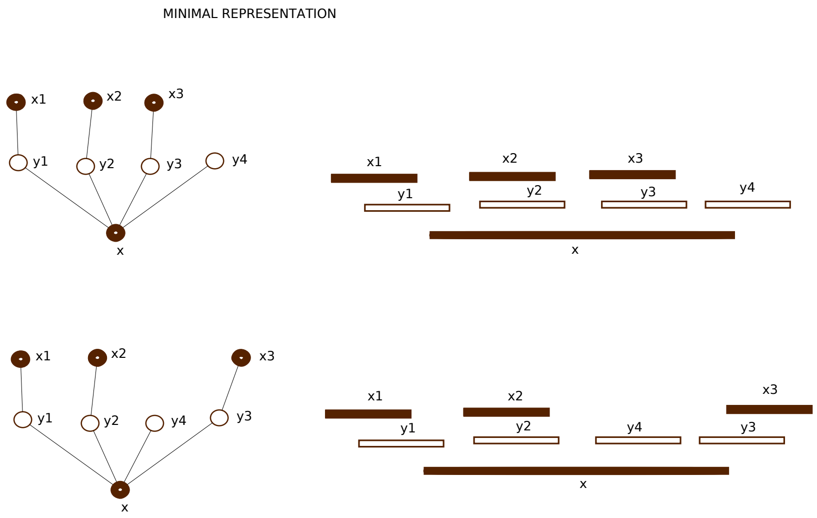


Figure 3.6: Impropriety of $imp(B_{\phi_1}) = 2$, $imp(B_{\phi_2}) = 1$

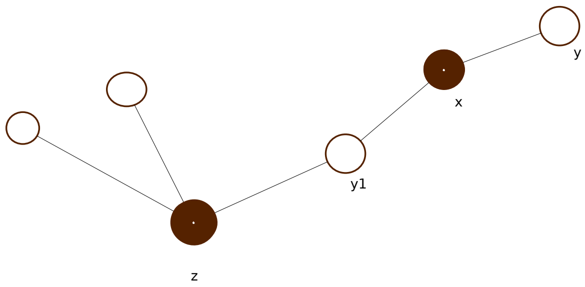


Figure 3.7: Exterior local components

Proof: We will prove this by contradiction. Let us assume that $B(X, Y)$ is a connected interval bigraph and $z \in X$ is a vertex in B with at least 3 exterior local components. So there exists 3 edges x_1y_1, x_2y_2, x_3y_3 in 3 of these exterior components such that $x_1, y_1, x_2, y_2, x_3, y_3 \notin N(z)$. We can find a path between any two pairs of these edges, say from x_1y_1 to x_2y_2 through z which avoids the neighborhood of x_3y_3 since $x_3, y_3 \notin N(z)$. Hence the edges x_1y_1, x_2y_2, x_3y_3 form an asteroidal triple of edges which is a forbidden subgraph for an interval bigraph [23]. Thus we get a contradiction. Hence any vertex z in an interval bigraph can have at most two exterior local components.

■

Note: Let $B(X,Y)$ be an interval bigraph. If C_i is an exterior component of any vertex $z \in X$ then in the interval representation there exist a vertex $y \in C_i \cap Y$ such that $I_y \cap I_z = \phi$. So C_i must always be on the side of I_z.

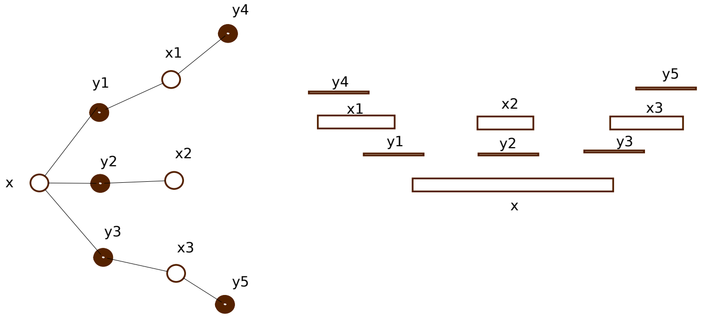


Figure 3.8: Interval bigraph with impropriety 1

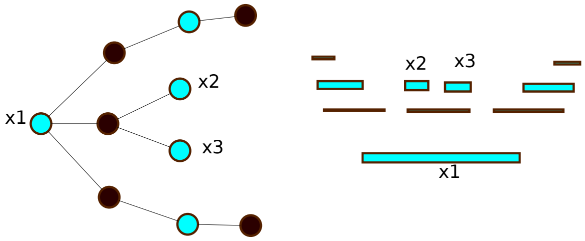


Figure 3.9: Interval bigraph with impropriety 2

The following definition and theorem leads to a bound on the impropriety.

Definition 3.2.7 *Let* $B(X,Y,E$ *be an interval bigraph. Let* $z \in X$ *and* $C_1, C_2, ..., C_n$ *be the local components of* z *ordered from left to right (the left hand components have intervals on the left hand side of the interval representation and the right hand components will have intervals on the right hand side of the interval representation) such that* C_1 *and* C_n *are chosen either to be exterior components if they exist or non-exterior local components such that* $|C_1 \cap X| \geq |C_i \cap X|, i = 2, ...(n-1)$ *and* $|C_n \cap X| \geq |C_i \cap X|, i = 2, ...(n-1)$*. The* weight of the vertex z *is the total number* *of vertices in* $(C_2 \cap X) \cup .... \cup (C_{n-1} \cap X)$*, which we will denote as* $wt(z)$*. We will call* $C_2, C_3, ..., C_{(n-1)}$ *the smaller non-exterior local components of* z*. The* weight *of* X *is the maximum of the weights of all the vertices* $x \in X$*, which we will denote as* $wt(X)$*. The weight of a vertex in* Y *and the weight of* Y *are defined similarly. The* weight of an interval bigraph, $wt(B)$*, is the maximum of the weights of its two partite sets.*

The graphs in Figure 3.9 and Figure 3.11 have weights 2 even though they have a different number of exterior components. This is because we consider the smallest non-exterior local components to calculate the weights.

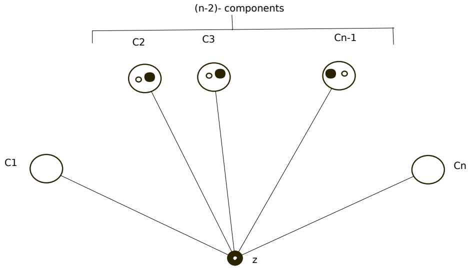


Figure 3.10: Calculation of weight

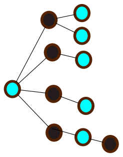


Figure 3.11: Interval bigraph with weight 2

The next theorem gives us a bound on the impropriety.

Theorem 3.2.8 *Let* $B(X, Y, E)$ *be an interval bigraph. Let* $z \in B$*. Then the impropriety of* B *is at least the weight of* z*.*

Proof: Let α be a minimal interval representation of the interval bigraph B. Let $z \in X$ and $C_1, C_2, ...C_n$ be the local components of z ordered from left to right as in Definition 3.2.7. Let I_z denote the interval of z. So $wt(z) = \sum_{i=2,...(n-1)} |C_i \cap X|$. The intervals of vertices from X in $C_2, ..., C_{n-1}$ must be inside I_z because there are vertices $y_1 \in C_1, y_n \in C_n$ that are not adjacent to any vertex from X in $C_2, ..., C_{n-1}$ but are adjacent to z. Hence, for any α and for any $z \in X$ $wt(z) \leq imp(z)$. Since $imp(B)$=max[$imp(z_i)$ for all $z_i \in B$], $imp(z) \leq imp(B)$, and so $wt(z) \leq imp(B)$. Hence the impropriety of B is at least the weight of z.

■

Corollary 3.2.9 *For any interval bigraph $B(X, Y, E)$, $imp(B) \geq wt(B)$.*

Proof: *The proof follows directly from the previous theorem and definitions.* ■

Note: Now that we have the above bound on weight, the reason behind the restriction of the containment to a single partite set can be discussed more explicitly. If we did not confine it to an individual partite set, the above result would fail to hold for interval bigraphs. For example, consider $K_{1,4}$. This is an interval bigraph. Let us call it $B(X, Y, E)$. Without loss of generality, there is only one vertex in X and the remaining four vertices belong to Y. By our definition of impropriety and weight, we have the $wt(X) = 0 = imp(B)$. Thus the above bound $wt(X) \leq imp(B)$ holds here. If we had not restricted the impropriety to a single partite set then the definition of weight would also be different. Weight also would not have been confined to an individual partite set. It would take into account all the vertices in the $n-2$ smallest non exterior components of any vertex with n local components. Therefore $imp(B) = 0$ and $wt(X) = 2$ which is illustrated in Figure 3.12. This defies the bound $wt(X) \leq imp(B)$. This bound holds for interval graphs and since it is easier to calculate $wt(B)$than $imp(B)$, we wanted this useful bound here.

3.3 p-critical interval bigraphs

We have seen in corollary 2.9 that the weight of any partite set of an interval bigraph can be at most the impropriety of the bigraph. Figure 3.13 and figure 3.14

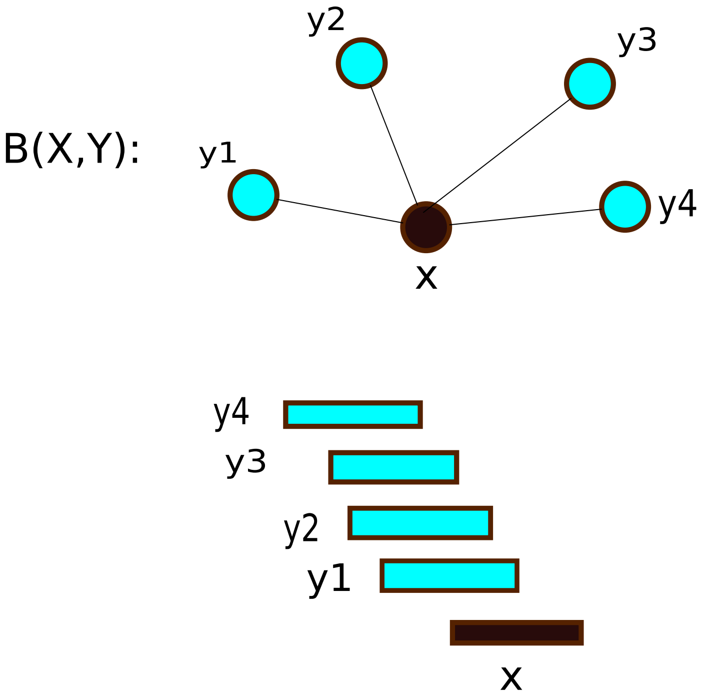


Figure 3.12: Changes in weight due to variation in definition

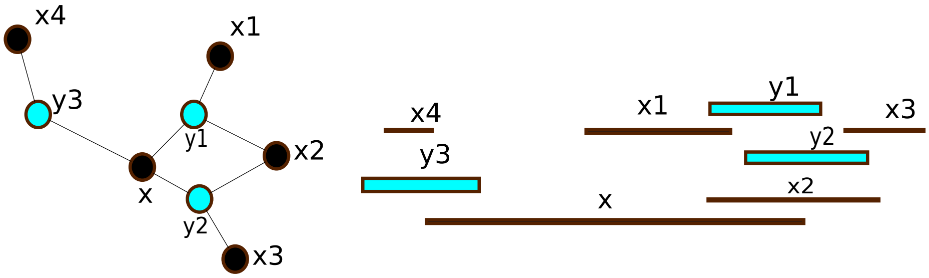


Figure 3.13: Interval bigraph where $wt(X) = 0 < imp(B) = 1$

demonstrate the two cases. We will look at the case when equality holds. The case when $imp(B) = wt(X)$ will lead us to the following definitions.

Definition 3.3.1 *Let $B(X, Y, E)$ be an interval bigraph. X is* balanced *if and only if $wt(X) = imp(B)$. If X is balanced, there exists a vertex $z \in X$ such that $wt(z) = imp(B)$ and such a vertex is called a* basepoint *of X.*

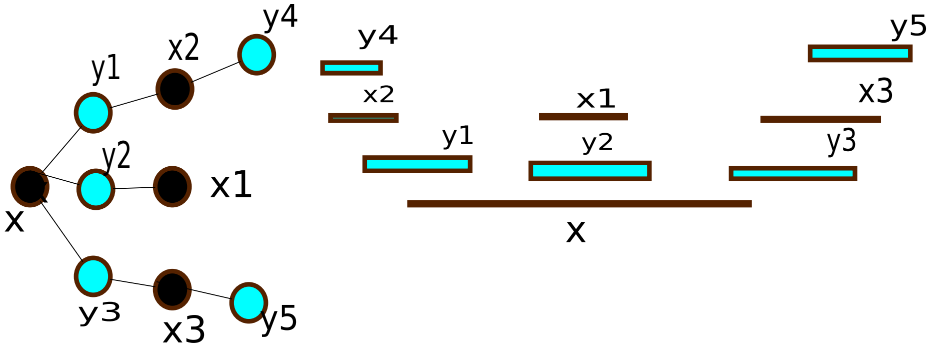


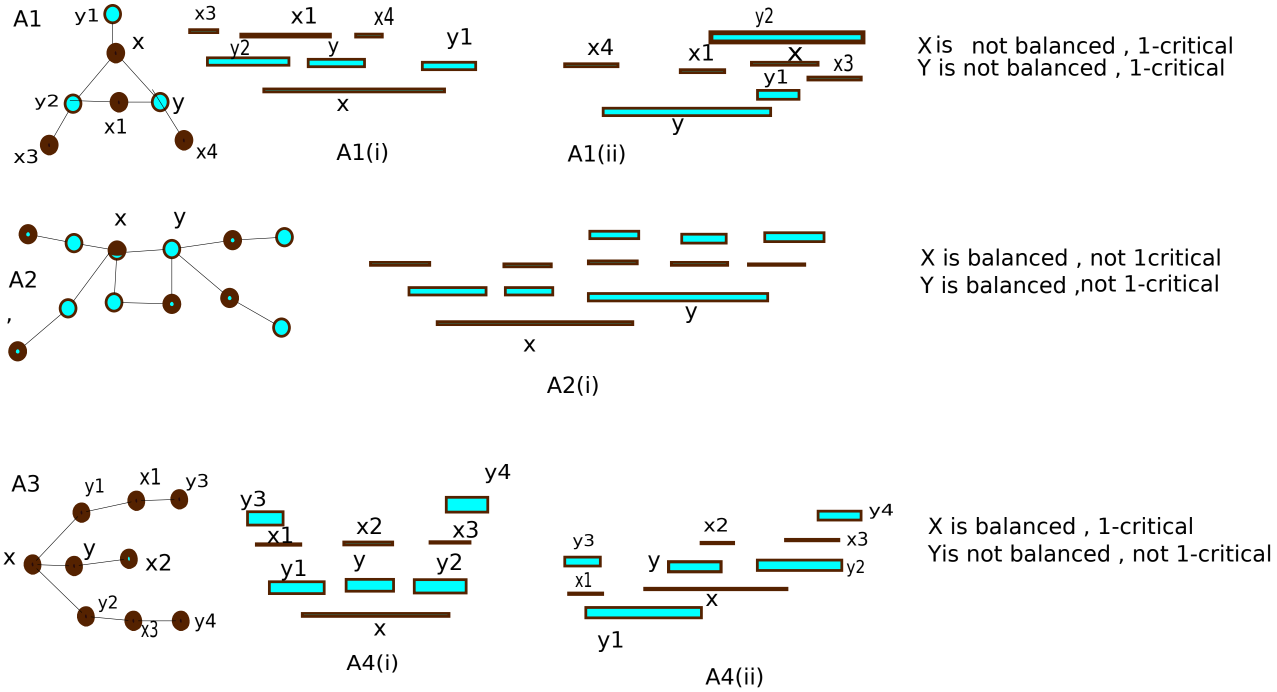
Figure 3.14: Interval bigraph where $wt(X) = 1 = imp(B) = 1$

Note that in any balanced partite set of an interval bigraph with $imp(B) > 0$ the basepoint must have at least 3 local components. Otherwise the weight becomes 0 and hence the impropriety is 0, leading to a contradiction. Moreover if the $wt(X) > 0$ then the basepoint of X must have at least 3 local components with vertices from X in all three components.

Definition 3.3.2 *A partite set X of an interval bigraph $B(X, Y, E)$ is p-*critical *with respect to impropriety if and only if B has impropriety p, and the removal of any vertex $z \in X$ decreases the impropriety of B.*

The balance and criticality of different partite sets might not be the same for an interval bigraph. In Figure 3.15 we have illustrations of balance and criticality of the partite sets of various interval bigraphs. We prove later in this section that both the partite sets of an interval bigraph cannot be balanced and p-critical at the same time.

We will now focus our attention on a special class of graphs. For the remaining part of this section we will consider interval bigraphs with a balanced and p-critical partite set. With this restriction on a partite set of an interval bigraph we will prove interesting results about the structures of this particular class of bigraphs. The next theorems and lemmas give us an idea about the structure of an interval bigraph with a balanced and p-critical partite set.



X is not balanced , 1-critical
Y is not balanced , 1-critical

X is balanced , not 1critical
Y is balanced ,not 1-critical

X is balanced , 1-critical
Yis not balanced , not 1-critical

Figure 3.15: Illustrations of balance and p-criticality

Theorem 3.3.3 *Let z be a basepoint of a balanced p-critical partite set X of the Bigraph $B(X,Y)$. If C is an exterior local component of z, then C has exactly one vertex from X.*

Proof: Let $B(X,Y)$ be an interval bigraph. Suppose X is balanced and p-critical and z is its basepoint. We will prove this by contradiction. We use techniques similar to those used by Beyerl and Jamison [3]. Let C be an exterior local component of z that has at least 2 vertices $x, v \in X$. Let x be the closest vertex to z such that it is on an edge xy and $y \notin N(z)$. Figure 3.16 illustrates possible positions of the vertex x. Let H be the graph obtained from $B(X,Y)$ by deleting all the vertices from C except those on the path from z to y that contains x. By theorem 3.2.8 $wt_H(z) \leq imp(H)$. Since X is p-critical, $imp(H) < imp(B)$. Again, since X is balanced $imp(B) = wt(X)$. Since z is a basepoint of X, $wt_X(z) = wt(X)$. Since the vertices removed from X are in the exterior component C, the smallest $(n-2)$

non-exterior components remain the same, which implies $wt_X(z) = wt_H(z)$.

Hence $wt_H(z) \leq imp(H) < imp(B) = wt(X) = wt_X(z) = wt_H(z)$, which is a contradiction.

■

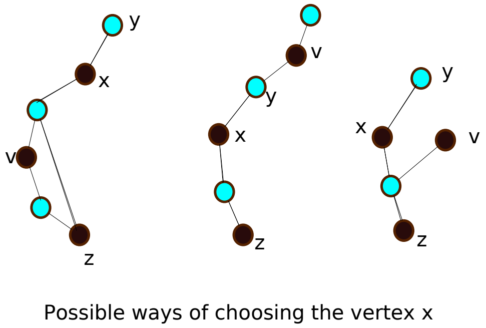


Figure 3.16: Choice

Next we show that if a partite set X is balanced and p-critical then it has a unique basepoint. Then we will prove several facts about interval bigraphs using the unique basepoint of either of its partite sets. It is easy to see that if the weight of any partite set X of an interval bigraph is 0 then a basepoint z has at most 2 local components with vertices from X. If the $wt(X) > 0$ then a basepoint has at least 3 local components with vertices from X in all three components. Depending on different scenarios, the local components can be at the side of the interval I_z or must be totally contained in it. In any interval representation, we will call the leftmost and rightmost end local components of any vertex its local side components. The remaining non-side local components will be called the local inner components. In Definition 3.2.7, C_1 and C_n are always the local side components. If they are exterior local components then they will contain vertices from Y non-adjacent to z and so they must be on the sides of I_z. If they are not exterior then in order to obtain a minimal representation they must again be on the two sides of I_z. We will call C_1 the leftmost local component and C_n the rightmost local component.

Lemma 3.3.4 *Any balanced and p-critical* $(p > 0)$ *partite set of an interval bigraph* $B(X, Y, E)$ *has exactly one basepoint.*

Proof: Let X be a balanced and p-critical partite set of an interval bigraph $B(X, Y, E)$. We need to show that X has only one basepoint. We will prove this by contradiction. Let us assume that X has two basepoints namely x and v. Since X is balanced and p-critical $(p > 0)$, both x and v must have at least 3 local components with vertices from X. Since x is a basepoint, it has at least 3 local components and v must be in one of them. This leads to the following cases:

Case1 : If $v \in$ C where C is an exterior local component of x, then, by Theorem 3.3.3, v is the only vertex from X in C. This implies that v can have exactly one local component containing vertices from X. This is a contradiction.

Case2 : If $v \in$ C where C is a non-exterior local inner component of x then every vertex from Y which is adjacent to v is also adjacent to x since C is not exterior. Thus B-v is connected. So v has just one local component which is a contradiction.

Case3 : If $v \in$ C where C is a non-exterior local side component of x, every vertex from Y which is adjacent to v is also adjacent to x because C is not exterior. We can conclude that v again has just one local component, which is a contradiction.

■

The following theorem proves that both the partite sets of an interval bigraph cannot be balanced and p-critical at the same time.

Theorem 3.3.5 *Let B(X,Y) be an interval bigraph. Both X and Y cannot be balanced and p-critical* $p > 0$ *at the same time.*

Proof:

We will prove this by contradiction. Let us assume that both X and Y are balanced and p-critical. Let v and w be the basepoints of X and Y respectively. Since X is balanced and p-critical and $p > 0$, its basepoint must have at least 3 local components with vertices from X. Let $C_1, C_2, ..C_j, ...C_n$ be n local components of v ordered from left to right and $n \geq 3$. Hence v is adjacent to at least n vertices from Y. Since B is connected, w must be in some local component of v. Let $w \in C_k, k \in 1, 2, ...n$. Since Y is balanced and p-critical, w must have at least 3 local components with one local component containing ($v \cup C_i$), $i = 1, ...n$ and $i \neq k$ as an induced subgraph, which has at least $(n-1)$ vertices from Y. Let us call this local component of w A_1 where $|(A_1 \cap Y)| \geq (n-1)$. Since $n \geq 3$ there are at least 2 local components of v say C_j, C_m and $j, m \neq k$, each of which contains a vertex from X not adjacent to w. Since C_j and C_m are contained in A_1, it implies that A_1 is an exterior component of w. Since A_1 is an exterior component of w it can contain exactly one vertex from Y [Theorem 3.3.3]. But $|(A_1 \cap Y)| \geq (n-1)$ and $n \geq 3$ which is a contradiction. Hence both X and Y cannot be balanced and p-critical at the same time.

■

Next we determine the general structure of an interval bigraph that has a balanced and p-critical partite set. The following lemma and theorem give an explicit idea about the cardinality of the local side components of the basepoint of any balanced and p-critical partite set of an interval bigraph. The lemma gives us the cardinality and the following theorem uses the lemma to demonstrate the structure. If a representation has been given, l_v and r_v will denote the left and right endpoints, respectively, of the interval I_v representing any vertex v.

Lemma 3.3.6 *Let X be a balanced and p-critical partite set of an interval bigraph $B(X,Y)$ and $v \in X$ be the basepoint of X with at most one exterior component. Let $C_1, C_2, ...C_n$ be local components of v ordered from left to right as in Definition 3.2.7. Let α be a minimal interval representation such that C_1 is a non-exterior local side component of v in α and C_n is the other local side component. If C_j is a local inner component of v such that $|C_j \cap X|$ is largest among all the other local inner components of v, then $|C_j \cap X| = |C_1 \cap X|$ if C_n is exterior and $|C_j \cap X| = |C_1 \cap X|$*

$= |C_n \cap X|$ *if* C_n *is non-exterior.*

Proof:

We will prove this by contradiction. Let $v \in X$ be the basepoint of X with the corresponding minimal interval representation α. The local components of v with respect to the minimal interval representation α are $C_1, C_2, ...C_n$ ordered from left to right with C_1 as the non- exterior local side component and C_n as the other local side component. First let C_n be an exterior local component and pick j such that $|C_j \cap X| \geq |C_i \cap X|$ for all $i = 2, 3, ...(n-1)$.

From Definition 3.2.7, $|C_1 \cap X| \geq |C_j \cap X|$. Assume that $|C_1 \cap X| > |C_j \cap X|$, then $|C_1 \cap X| > |C_i \cap X|, i = 2, 3, ...(n-1)$. Let $x_1 \in C_1 \cap X$ and look at $B - x_1$. Since C_1 is non-exterior, every vertex from Y in C_1 adjacent to x_1 is adjacent to v. So $B - x_1$ is still connected. Since $|C_1 \cap X| > |C_i \cap X|, i = 2, 3, ...(n-1)$, removal of x_1 from C_1 does not affect the weight of X. Hence $wt(X - x_1) = wt(X)$. Since X is p-critical, removal of any vertex from X will reduce the impropriety of B by one. Thus $imp(B - x_1) < imp(B)$ and X being balanced implies $imp(B) = wt(X)$. So we have the following relation:

$imp(B - x_1) < imp(B) = wt(X) = wt(X - x_1)$.

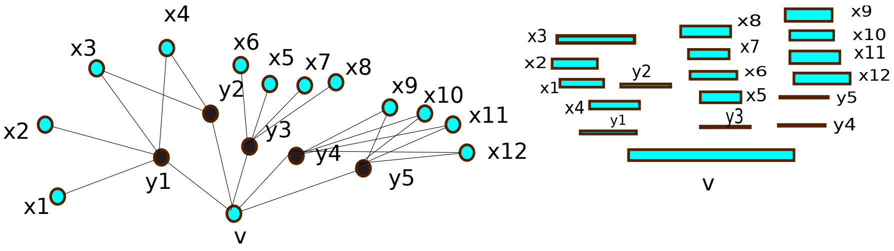
Thus $imp(B - x_1) < wt(B - x_1)$, which is a contradiction. Hence $|C_j \cap X| = |C_1 \cap X|$.

If C_n is also a non-exterior local side component then by the same argument we can prove that $|C_j \cap X| = |C_n \cap X|$. Hence $|C_j \cap X| = |C_n \cap X| = |C_1 \cap X|$.

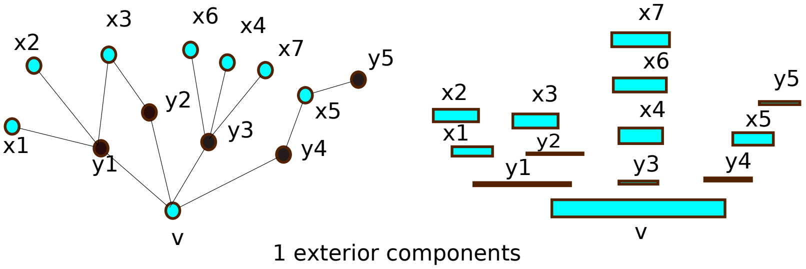
■

The next theorem illustrates the structure of an interval bigraph B with a balanced and p-critical partite set where $p > 0$. It proves that B can have exactly three structures depending on the number of exterior components.

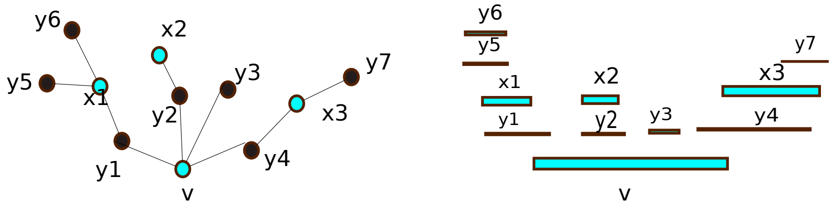
Theorem 3.3.7 *Let* $B(X, Y, E)$ *be an interval bigraph. If* X *is balanced and* p*-critical*



0 exterior component



1 exterior components



2 exterior components

Figure 3.17: Possible structures of an interval bigraph with a balanced and p-critical partite set

then B *has one of the following structures:*

i) $B \simeq B_1$ *where* $B_1(X, Y)$ *is an interval bigraph and* v *is the basepoint of* X *which has* n *local non-exterior components* $C_1, C_2, ..C_i, ...C_n$ *ordered from left to right as in Definition 3.2.7 such that there are at least 3 local components* C_1, C_j, C_n *where* $|C_j \cap X| = |C_1 \cap X| = |C_n \cap X| \geq |C_i \cap X|, i = 2, ..., (n-1)$*. Furthermore each of* C_1 *and* C_n *has a vertex from* Y *which is adjacent to all vertices from* X *in* C_1 *and* C_n *.*

ii) $B \simeq B_2$ *where* $B_2(X, Y)$ *is an interval bigraph and* v *is the basepoint of* X *which has exactly one exterior component* C_n *and* $(n-1)$ *local non-exterior components* $C_1, C_2, ..C_i, ...C_{n-1}$ *ordered from left to right as in Definition 3.2.7 such that it has at least 2 local components* C_1, C_j *where* $|C_j \cap X| = |C_1 \cap X| \geq |C_i \cap X|, i = 2, ...(n-1)$*. Furthermore the left most local component* C_1 *has a vertex from* Y *which is adjacent to all vertices from* X *in* C_1*.*

iii)$B \simeq B_3$ *where* $B_3(X, Y)$ *is an interval bigraph and* v *is the basepoint of* X *which has* $C_1, C_2, ..C_i, ...C_n$ *local components ordered from left to right with exactly two exterior components* C_1, C_n *and the remaining* $(n-2)$ *components are local inner components.*

Proof: $B(X, Y, E)$ is an interval bigraph where X is balanced and p-critical. We need to prove that B is isomorphic to B_1, B_2 or B_3 depending on the number of exterior components the basepoint of X has. Let $v \in X$ be the basepoint of X with n local components $C_1, C_2, ..C_i, ...C_n$ ordered from left to right. By Lemma 3.2.6 we know that v can have at most 2 exterior components and this gives us 3 cases.

Case I: If $v \in X$ has no exterior components then all the components of v are non-exterior local components. Hence by Lemma 3.3.6 there must be at least 3 components C_1, C_j, C_n where $|C_j \cap X| = |C_1 \cap X| = |C_n \cap X| \geq |C_i \cap X|, i = 2, ...(n-1)$. Thus $B \simeq B_1$.

Let us now analyze the local side component C_1. Since C_1 is not an exterior component of $v \in X$, it can have more than one vertex from X. Since the weight of v is determined by the smallest $(n-2)$ local inner components of v and by the previous Lemma $|C_1 \cap X| = |C_j \cap X| \geq |C_i \cap X|$ for all $i = 2, 3, ...(n-1)$, the component C_1 does not contribute anything towards the weight of v. Let us assume that $|C_j \cap X| = k$. Hence $|C_1 \cap X| = k$. Since X is balanced, p-critical with $p > 0$, C_1 must have vertices from both X and Y. Let $x_1, x_2, ...x_k \in C_1$. There can be more than one vertex from Y in C_1. Since C_1 is a non-exterior component, $I_{y_j} \cap I_v \neq \phi$ for all $y_j \in C_1$ (*).

Since X is balanced, no x_i contributes to the impropriety of v for $x_i \in C_1$, which implies that none of the $x_i, i = 1, 2, ...n$ are contained in I_v(**).

Since C_1 is the leftmost local component, it implies that

$$l_{x_i} < l_v \text{ from(*) for all } i$$

$$r_{y_j} \geqslant l_v. \text{ from(**) for all } j$$

Take the x_i such that r_{x_i} is the least. This x_i must be adjacent to some y_j which is adjacent to v. Thus $I_{y_j} \cap I_{x_i} \neq \phi$, and by our choice of x_i, $I_{y_j} \cap I_{x_m} \neq \phi$ for all $x_m \in C_1$. So y_j is adjacent to all vertices from X in C_1. This structure is illustrated in Figure 3.17. Similarly C_n also has a vertex from Y which is adjacent to all vertices from X in C_n.

Case II: If $v \in X$ has exactly 1 exterior component then it has a non-exterior side component. Let C_n be the exterior component. Again by Lemma 3.3.6 it must have at least 2 local components C_j and C_1 such that $|C_1 \cap X| = |C_j \cap X| \geq |C_i \cap X|$ for all $i = 2...., (n-1)$ and one of these, say C_1, must be the local side component. Thus $B \simeq B_2$. By arguments similar to those used in Case I, C_1 also has a vertex from Y which is adjacent to all vertices from X in C_1. This is illustrated in Figure 3.17.

Case III: If $v \in X$ has 2 exterior components C_1 and C_n then all the other $(n-2)$ local components of v are non-exterior local components. The exterior local components cannot be local inner components otherwise we would not be able to obtain any interval representation of the bigraph. Hence the two exterior components C_1, C_n must be the two side components of v and the remaining $(n-2)$ components are the local inner components. Thus $B \simeq B_3$. This structure is also illustrated in Figure 3.17.

■

3.4 Conclusions and future work

We investigated the structures and characteristics of p-improper interval bigraphs. The same ideas could be naturally extended to interval k-graphs. These are graphs with a proper coloring where each vertex v can be assigned an interval I_v of the real line such that two vertices are adjacent if and only if their corresponding intervals overlap and each vertex has a different color. It would be interesting to study the roles played by impropriety and criticality on interval k-graphs.

4. Characterization of unit probe interval 2-trees

4.1 Introduction

Interval graphs were introduced by Hajos [22], and were then characterized by the absence of induced cycles of length larger than 3 and asteroidal triples by Lekkerkerker and Boland [26] in 1962. Unit interval graphs are the interval graphs that have an interval representation in which each interval has unit length. In 1969, Roberts [36] proved that the classes of proper interval graphs and unit interval graphs coincide and he showed that interval graphs that have no induced $K_{1,3}$ are unit interval graphs. Much more recently in 2007 Gardi gave a much shorter new constructive proof of Roberts original characterization of unit interval graphs [18].

A graph is a probe interval graph if there is a partition of $V(G)$ into sets P and N and a collection $\{I_v : v \in V(G)\}$ of intervals of R such that, for $u, v \in V(G)$, $uv \in E(G)$ if and only if $I_u \cap I_v \neq \phi$ and at least one of u or v belongs to P. The sets P and N are called the probes and nonprobes, respectively. If, for G a probe interval graph, the members of $I_v : v \in V(G)$ are closed intervals of identical length, then G is a unit probe interval graph. The probe interval graph model was invented in connection with the task called physical mapping used in connection with the human genome project by Zhang and Zhang et. al. [43], [44].

One way to describe the structure of a class of graphs is by finding its complete list of minimal forbidden induced subgraphs; No complete forbidden induced subgraph characterization for general probe interval graphs has been found so far. As the characterization of probe interval graphs seems to be difficult, research has focussed on classes of probe interval graphs.

Family of 2-trees are the set of all graphs that can be obtained by the following construction: (i) the 2-complete graph, K_2, is a 2-tree; (ii) to a 2-tree Q' with $n-1$ vertices ($n > 2$) add a new vertex adjacent to a 2-complete subgraph of Q'. Li Sheng first characterized cycle-free probe interval graphs [41]. As a natural extension of the characterization for trees, Pržulj and Corneil attempted a forbidden subgraph characterization of 2-trees that are probe interval graphs and they found at least 62 distinct minimal forbidden induced subgraphs for probe interval graphs that are 2-trees [34]. More recently Brown, Flesch and Lundgren extended the list to 69 and gave a characterization in terms of sparse spiny interior 2-lobsters [7]. In 2009 Brown, Sheng and Lundgren gave a characterization of cycle-free unit probe interval graphs [15]. In [11] various characterizations are given for the probe interval graphs that are bipartite. These characterizations are for classes of bipartite graphs, but no general characterization of bipartite probe interval graphs were given there. However recently, Brown and Langley characterized unit probe interval graphs for bipartite graphs [10]. This chapter restricts to the unit case of probe interval graphs which are 2-trees.

In this chapter we characterize 2-trees that are unit probe interval graphs. In Section 2 we introduce some important subclasses of 2-trees. In addition, we introduce several basic results that will be useful in the characterization. In the Section 3 we characterize 2-caterpillars and interior 2-caterpillars in terms of forbidden induced subgraphs and show that 2-trees that are unit probe interval graphs have to be interior 2-caterpillars. While this significantly reduces our focus, it turns out the problem remains very difficult. A major reason for this is that even determining which 2-paths are unit probe interval graphs is difficult. So in Section 4 we give a list of forbidden subgraphs for 2-paths that are unit probe interval graphs. Then in Section 5 we complete the characterization for 2-paths which are unit probe interval graphs. In Section 6 we extend the list of forbidden subgraphs from section 4 to include the remaining forbidden subgraphs for interior 2-caterpillars. In Section 7 we complete the list of forbidden subgraphs for 2-tree unit probe interval graphs using 27 subgraphs. The size of the list illustrates the difficulty of this problem.

4.2 Preliminaries

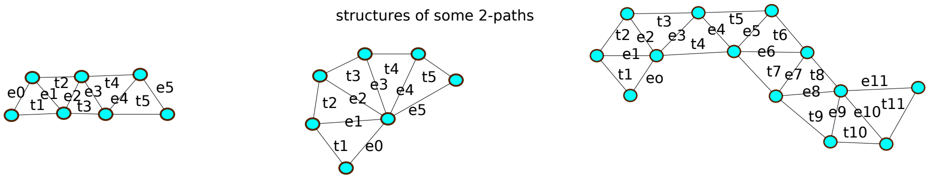


Figure 4.1: Examples of some 2-paths

In this section we introduce some subclasses of 2-trees. Also we give several basic results that will be useful throughout the chapter. To describe the structure of 2-trees, we use the idea of a 2-path introduced by Beineke and Pippert in [2].

Definition 4.2.1 *[2] A* 2*-path is an alternating sequence of distinct* 2 *and* 3*-cliques,* $(e_o, t_1, e_1, t_2, e_2, ..., t_p, e_p)$*, starting and ending with a* 2*-clique and such that* t_i *contains exactly two distinct* 2*-cliques* e_{i-1} *and* e_i $(1 \leq i \leq p)$*. The* length *of* 2*-path is the number,* p*, of* 3*-cliques.*

In general 2-paths are much more complex than paths. As we will see later that characterization for 2-paths which are unit probe interval graphs is very difficult.

Drawing the connection to tree further, Proskurowski introduced the notion of a 2-caterpillar in [32]. A 2-leaf is a vertex whose neighborhood is a 2-clique.

Definition 4.2.2 *[32] A* 2-caterpillar P *is a* 2*-tree in which the deletion of all* 2*-leaves results in a* 2*-path, called the body of* P*. A* 2*-caterpillar* P *is an* interior 2-caterpillar *if for any* 2*-leaf* v*,* v *is adjacent to all vertices of some* 2*-complete subgraph* e_i *of any longest* 2*-path of* P*.*

In the Section 3 we will obtain a characterization of 2-caterpillars and interior 2-caterpillars.

Definition 4.2.3 *The* 2-distance *between a* 2*-leaf and a longest* 2*-path of* G *is the length of the shortest* 2*-path between them.*

Definition 4.2.4 *An* asteroidal triple *(AT) in* G *is a set* A *of three vertices such*

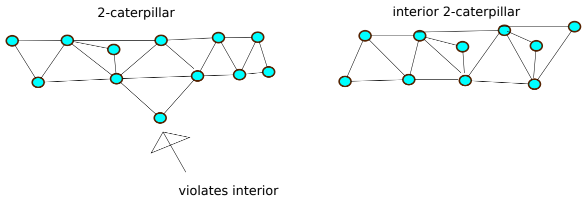


Figure 4.2: On the left is a 2-caterpillar, and on the right is an interior 2-caterpillar

that between any two vertices in A there is a path that avoids all neighbors of the third.

It is well-known that interval graphs cannot contain asteroidal triples. However, probe interval graphs can contain asteroidal triples provided the restrictions in the following lemmas are obeyed. The following lemmas will also be very useful in proving some of the results in the present and forthcoming sections specially in showing that certain 2-trees that are probe interval graphs are not unit probe interval graphs.

Lemma 4.2.5 *[41] At least one vertex in an AT of a probe interval graph must be a non probe.*

Lemma 4.2.6 *[41] In every AT there must exist a non-probe vertex u such that there exist a path between the other two vertices in the AT that has a non-probe internal vertex.*

Lemma 4.2.7 *[6] The vertex of degree 3 in any induced* $K_{1,3}$ *of a unit probe interval graph must be a probe and at least two of the vertices of degree 1 of the induced* $K_{1,3}$ *must be non-probes.*

Definition 4.2.8 *A k*-fan *is a 2-path of length k such that all* e_i*'s are incident to a common vertex which we call the center; all other vertices are called the radial vertices.*

Definition 4.2.9 *4-fan is a 2-tree made up of four 2-cliques such that all the cliques share one particular vertex called the central vertex as in Figure 4.3. The vertices* *along the circumference of the 4-fan are called the radial vertices. Radial vertices with degree 2 are the* end radial vertices. *The vertex with degree 3 which is equidistant from both the end radial vertices will be called the* central vertex *or* mid radial vertex.

Definition 4.2.10 *The* Merge *of two 2-trees T_i and T_j takes place if the intersection of their vertex set is not null. So $V(T_i) \cap V(T_j) \neq \phi$ where $V(T_i)$ and $V(T_j)$ denotes the vertex sets of T_i and T_j respectively.*

For convenience, we will introduce some notation. From now on the right end point and left end point of the interval corresponding to a vertex v will be denoted by $r(I_v)$ and $l(I_v)$ respectively. By $u \leftrightarrow v$ we will mean adjacency between u and v. We will also use the acronym UPIG for unit probe interval graph.

We will name a 4-fan by a list of six vertices, say $(a - b - c - d - e - f)$ with center at c by which we mean (a, b, d, e, f) is the path of radial vertices and c is the center of the 4-fan.

Lemma 4.2.11 *The center of a 4-fan must be a probe, the mid-radial (or the central vertex) vertex and at least one of the end-radial vertices must be non-probes; the other two vertices must be probes.*

Proof:

Let $F = (\{\ c, r_1, r_2, r_3, r_4, r_5\ \},$E) be a 4-fan where $r_1, \ldots.., r_5$, in order, from the path which contains the radial vertices and c is the center vertex. By Brown, Sheng, Lundgren, c is a probe and at most one of $\{\ r_1, r_3, r_5\ \}$ is a probe and hence r_2 and r_4 must be probes. Now observe the representation $I_{r_1} = I_{r_2} = [0, 1]$, $I_c = I_{r_3} = [1, 2]$, $I_{r_4} = I_{r_5} = [2, 3]$, works with r_1, r_5 as non-probes; if r_5 is a probe (or r_1 is a probe), then $I_{r_3} = [0.5, 1.5] (I_{r_3} = [1.5, 2.5])$ can be used.

■

A 3-sun, denoted F1 and depicted in the Figure 4.6 is a 2-caterpillar formed by three 3-cliques with a 2-leaf on the non-interior edge of the 3-clique at the middle. It

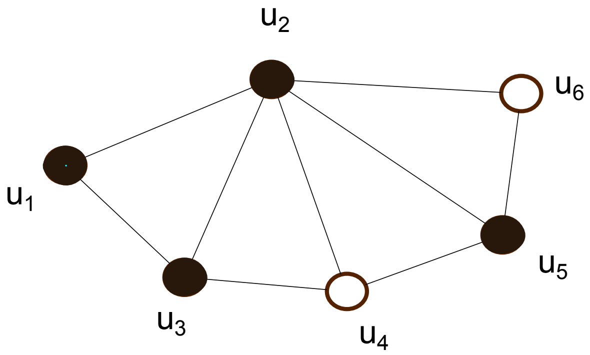


Figure 4.3: 4-fan

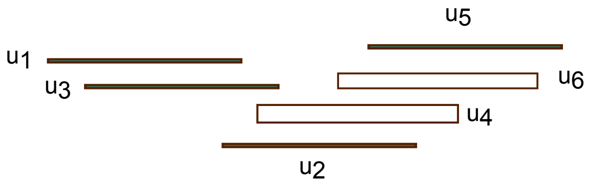


Figure 4.4: Unit probe interval representation of a 4-fan

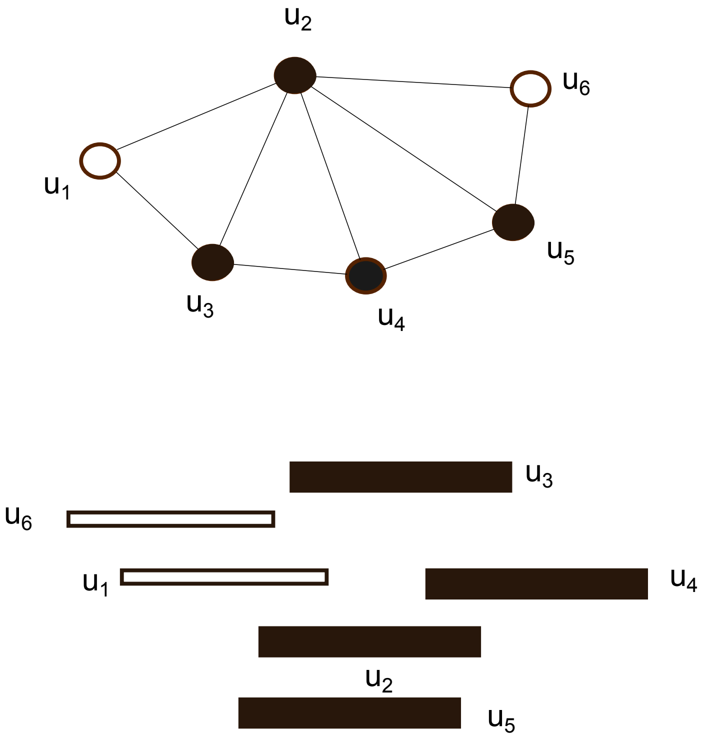


Figure 4.5: An illustration where unit probe interval representation of a 4-fan fails to work

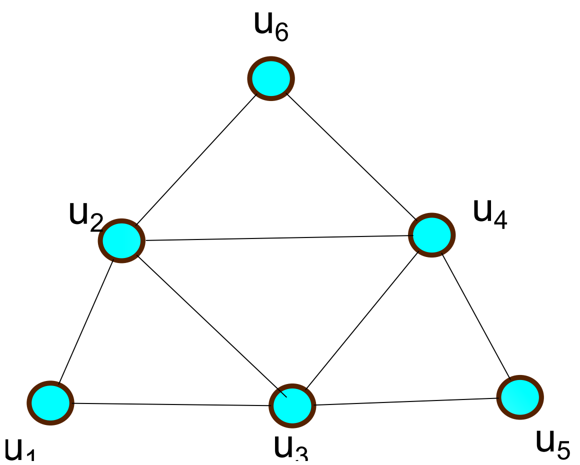


Figure 4.6: F1

is a well-known fact that the 3-sun is not an interval graph. It is easy to show that it is a probe interval graph. In the next lemma we will prove that it is not a unit probe interval graph.

Lemma 4.2.12 *The 2-tree F1 (3-sun) is not a unit probe interval graph.*

Proof:

Let us assume that $F1$ is a UPIG. It can be observed from Figure 4.7 that $F1$ is a probe interval graph. Without loss of generality, by Lemma 4.2.5, let u_6 be a non probe. Since u_2 and u_4 are adjacent to u_6, so they are probes. By Lemma 4.2.6, u_3 must be a non probe. Hence u_1 and u_5 are probes. We now attempt to draw the unit probe representation of F1. We know that $I_{u_3} \cap I_{u_1} \neq \phi$ and $I_{u_3} \cap I_{u_5} \neq \phi$. We make the intervals of u_1 and u_5 as far apart as possible. Thus I_{u_3} is flanked on both sides by $I_{u_1}$and I_{u_5}. I_{u_2} and I_{u_4} are drawn such that $r(I_{u_1}) < r(I_{u_2})$ and $r(I_{u_4}) < r(I_{u_5})$ as shown in Figure 4.7. u_6 is adjacent to u_2 and u_4 and the intervals of u_2 and u_4 overlap somewhere along the middle of the interval of u_3 as shown in Figure 4.7. Hence the length of the interval of u_6 must be reduced so that it can avoid overlapping with the intervals of u_1 and u_5 which is a contradiction. Therefore, $F1$ is not a unit probe

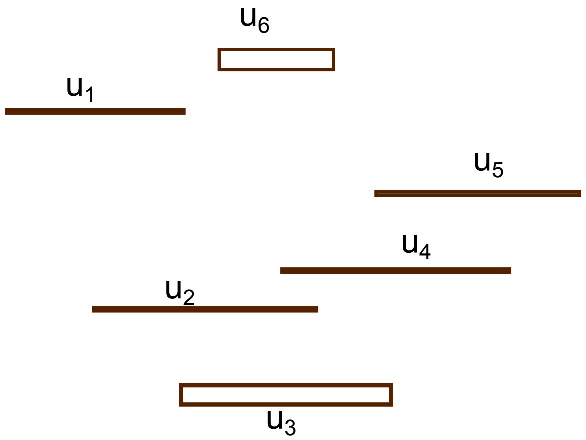


Figure 4.7: Probe interval representation of a 3-sun or F1

interval graph.

■

We will now prove that probe interval graph $E1$ shown in figure 4.8 is not a unit probe interval graph. We will later use these two results ($F1$ and $E1$ do not have unit probe interval representations) to characterize interior 2-caterpillars.

Lemma 4.2.13 *The 2-tree probe interval graph E1 shown in Figure 4.8 is not a unit probe interval graph.*

Proof: We know that $E1$ is a probe interval graph. Let us assume that $E1$ has a unit probe interval representation. Vertices u_2, u_1, u_3, u_4, u_5 and u_6 form a 4-fan as in Figure 4.8. The vertex u_2 is the center of the 4-fan and so it must be a probe. The vertex u_4 is the central radial vertex of the same 4-fan and so it must be a non-probe. As seen in Figure that u_4, u_3, v_1 and u_5 form an induced $K_{1,3}$ with center at u_4 and so u_4 must be a probe which is a contradiction. Hence $E1$ is not a unit probe interval graph.

■

4.3 Characterization of 2-caterpillars

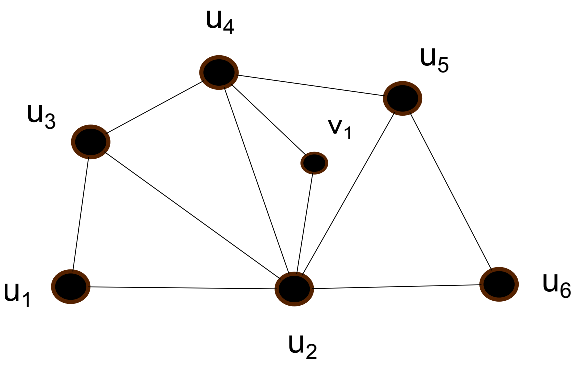


Figure 4.8: E1

In this section we significantly reduce the class of 2-trees we need to consider for unit probe interval graphs. First we reduce the class of 2-tree unit probe interval graphs to 2-caterpillars and then to interior 2-caterpillars. Thus in this section we give a complete characterization of 2-caterpillars and interior 2-caterpillars. We will see that the complete list of forbidden subgraphs for 2-trees which are 2-caterpillars consists of 4 subgraphs. Furthermore we also prove that all 2-trees which are unit probe interval graphs must be interior 2-caterpillars. This result significantly reduces our spectrum from a wide range of graphs to a relatively small class of interior 2-caterpillars.

Theorem 4.3.1 *A 2-tree is a 2-caterpillar if and only if it does not contain B1, B1', B2, B3 as induced subgraphs.*

Proof: Let G be a 2-tree which is a 2-caterpillar. By the definition of 2-caterpillar, G is a 2-tree such that after deletion of all 2-leaves of G we get a 2-path, called the body of P. If G= $B1, B1', B2, B3$ then G - {2-leaves of G } is not a 2-path as seen in Figure 4.9 and so G cannot contain them as induced subgraphs.

Let G be a 2-tree without $B1, B1', B2, B3$ as induced subgraphs. Suppose G is not a 2-caterpillar. Let $(e_o, t_1, e_1, t_2, e_2, ..., t_p, e_p)$ be a longest 2-path of G and label it P.

Since G is not a 2-caterpillar, G - {2-leaves of G } is not a 2-path. If every 2-leaf of G were a 2-leaf at distance 1 from P then G - { 2-leaves } would be a 2-path which is a contradiction. Hence G must have a 2-leaf at 2-distance 2 or more from P and this can happen in two ways. G has a 2-path of length at least 2 originating from some e_i or it has a 2-path of length at least 2 originating from a pair of vertices of some t_i not equal to e_{i-1} or e_i

Case(1): Assume that G has a 2-path of length at least 2 originating from some e_i. Since P is assumed to be a longest 2-path of G, there exists a 2-path of length at least 2 on either side of e_i on P. Let X be a 2-path of length 2 which originates from 2-clique e_i. Therefore the smallest structure satisfying these conditions is a 2-path P of length 4 with X starting from the middle 2-clique, e_i and its length is 2. According to Pržulj and Corneil [34], there are 2 non-isomorphic 2-paths of length 4 which are A_4^1 and A_4^2 as in Figure 4.10. The first vertex x_1 of X has 2 choices of positions as in Figure 4.11 and the second vertex x_2 of X has 2 choices again. Therefore we get four 2-tree subgraphs A_i, $i = 1, 2, 3, 4$ as in Figure 4.11. Observing that A_i, $i = 1, 2, 3$, are isomorphic to $B1$ and A_4 is isomorphic to $B1'$, it follows that G cannot contain these 2-paths. Hence G cannot have 2-paths of length 2 or more originating from some interior edge e_i.

Case(2): Now let us assume that G has a 2-path X of length at least 2 originating from a pair of vertices of some t_i not equal to e_{i-1} or e_i. Since P is assumed to be a longest 2-path of G, there exists a 2-path of length at least 2 on either side of t_i on P. Let X be a 2-path of length 2 which originates from a pair of vertices of some t_i not equal to e_{i-1} or e_i. Therefore the smallest structure satisfying these conditions is a 2-path P of length 5 with X starting from the middle clique, t_i and its length is 3. According to Pržulj and Corneil [34], there are three non-isomorphic 2-paths of length 5 which are A_5^1, A_5^2, A_5^3 given in Figure 4.12. Since t_i has just one edge other than e_{i-1}, e_i, there is exactly one way to add the first vertex x_1 of X. Let x_2

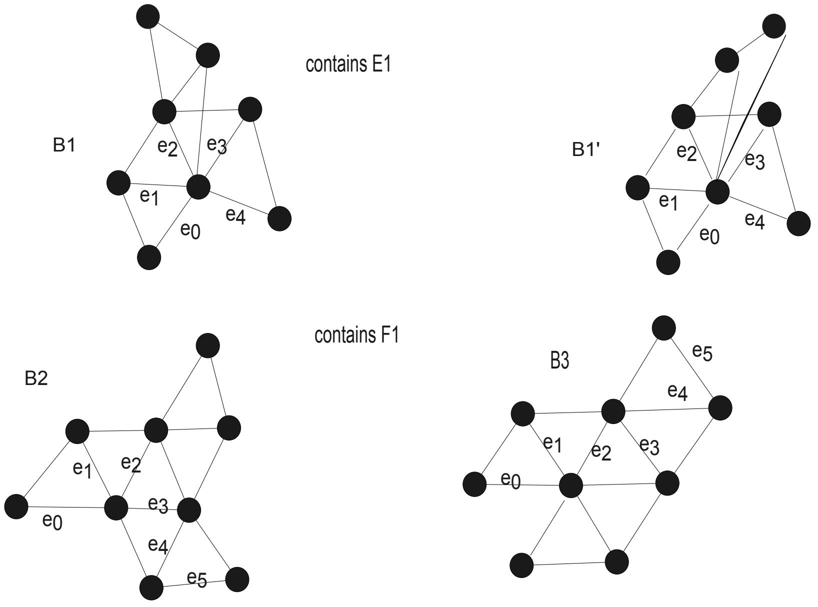


Figure 4.9: Forbidden subgraphs for 2-caterpillar

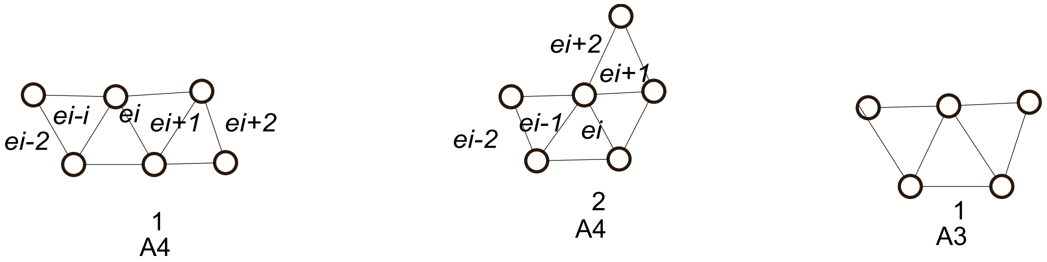


Figure 4.10: Construction 1

be the second vertex of X. The vertices adjacent to x_2 are x_1 and one of the vertices from $N(x_1)$. Thus we get six 2-trees $A_i, i = 1, 2, 3, ..., 6$ as in Figure 4.13. Observe that A_1, A_2, A_3, A_4 and A_5 are isomorphic to $B3$ and A_6 is isomorphic to $B2$. But G cannot have $B2, B3$ as induced subgraphs. Hence G cannot have any 2-path of length at least 2 originating from a pair of vertices of some t_i not equal to e_{i-1} or e_i. Hence from cases (1) and (2) we can conclude that G cannot have any 2-leaf at 2-distance 2 or more. So G must be a 2-caterpillar.

■

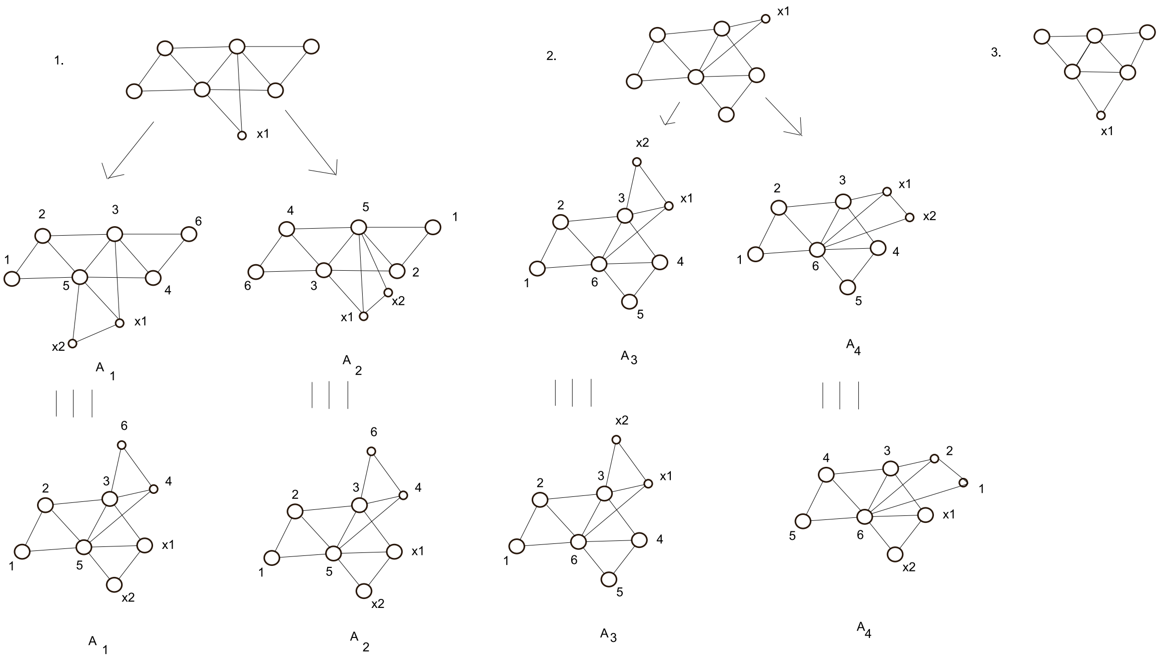


Figure 4.11: Construction 2

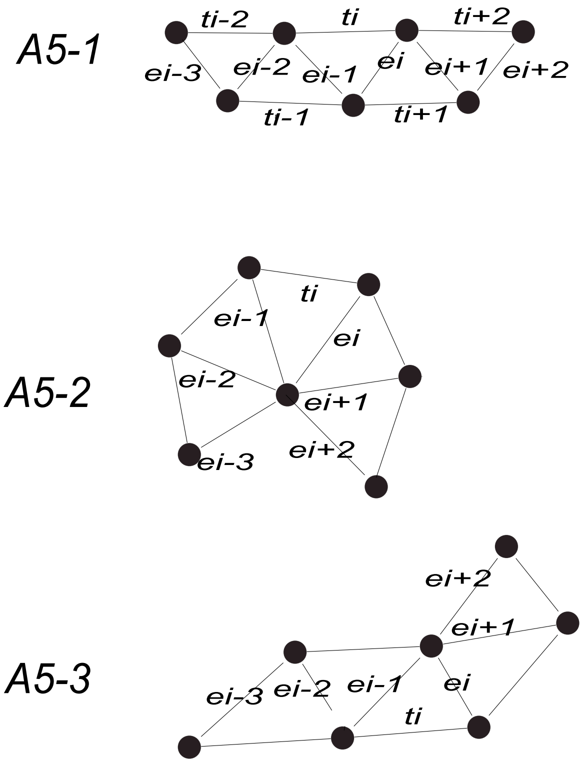


Figure 4.12: Construction 3

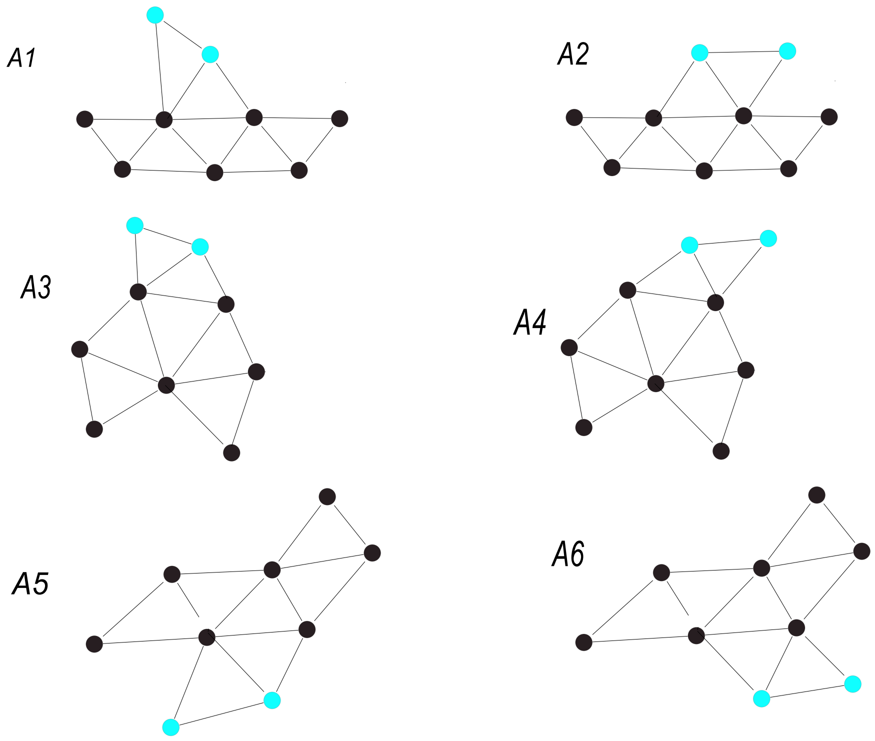


Figure 4.13: Six 2-trees called A_i's

Theorem 4.3.2 *A* 2*-caterpillar is an interior* 2*-caterpillar if and only if it does not contain F1 as an induced subgraph.*

Proof: Let G be a 2-tree which is a 2-caterpillar with $[e_0, t_1, ...e_n]$, a longest 2-path of G which we call P. We first assume that it is an interior 2-caterpillar. So it cannot have any 2-leaf adjacent to the end points of a non-interior edge. Since $F1$ is a 2-caterpillar with a longest 2-path being $e_0, t_1, e_1, t_2, e_2, t_3, e_3$ and a 2-leaf originating from a pair of vertices of t_2 not equal to e_1 or e_2, it is a non-interior 2-caterpillar. Hence G cannot have $F1$ in it as an induced subgraph.

Now we assume G is a 2-caterpillar without $F1$. If G is not interior then it has a 2-path of length 1 originating from a pair of vertices of some $t_i \neq e_{i-1}, e_i$. Since P is assumed to be a longest 2-path of G, there exists a 2-path of length at least 1 on either side of t_i on P. Let X be a 2-path of length 1 which originates from a pair of vertices of some t_i not equal to e_{i-1} or e_i. Therefore the smallest structure satisfying these conditions is a 2-path P of length 3 with X starting from the middle clique, t_2 and its length is 1. According to Pržulj and Corneil [34], there is just one 2-path of length 3 which is A_3^1 depicted in Figure 4.10. Thus the vertex x_1 that forms X is adjacent to a pair of vertices in t_2 such that $x_1 \not\leftrightarrow e_1, e_2$. Thus x_1 can be added to P in just one way, as in number 3 of Figure 4.11 which forms $F1$. This is a contradiction since G does not contain any subgraph isomorphic to $F1$. Hence G cannot have a 2-leaf at a non-interior edge. So G must be an interior 2-caterpillar.

■

Theorem 4.3.3 *A 2 tree UPIG G is an interior 2-caterpillar*

Proof: Let G be 2-tree unit probe interval graph. If G is not a 2-caterpillar then, by theorem 4.3.1, it must containone of $B1, B1', B2$, or $B3$ as an induced subgraph. Furthermore as seen in Figure 4.9 $B1$ and $B1'$ contains $E1$, $B2$ and $B3$ contains $F1$ as induced subgraphs. Hence G should also contain $E1$ or F1 as induced subgraphs provided G is not a 2-caterpillar which is a contradiction since $E1$ and $F1$ are forbidden subgraphs for unit probe interval graphs. Hence G cannot contain $B1, B1', B2$ or $B3$ as induced subgraphs. So by Theorem 4.3.1 G must be a 2-caterpillar.

We have proved that any 2-caterpillar is interior if and only if it does not contain $F1$ as an induced subgraph. But $F1$ is a forbidden subgraph for unit probe interval graphs. Hence G cannot contain $F1$ as an induced subgraph and so it must be an interior 2-caterpillar. ■

4.4 Forbidden subgraphs for 2-paths which are unit probe interval graphs

Interior 2-caterpillars can be thought of as 2-paths with 2-leaves on some of the interior edges. So when characterizing interior 2-caterpillars which are unit probe interval graphs the natural step is to characterize 2-paths which are unit probe interval graphs. Once we have a complete characterization of the 2-path unit probe interval graphs we will keep adding 2-leaves at every possible interior edge to determine which structures fail to be a unit probe interval graphs thus deriving a list of forbidden subgraphs for interior 2-caterpillars which are unit probe interval graphs. As we have mentioned earlier the characterization problem for 2-paths in itself turns out to be very difficult. In this section we give a list of forbidden subgraphs for 2-paths which are unit probe interval graphs. The forbidden subgraphs are named as $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$ and $F11$. All of these subgraphs are probe interval graphs but they fail to have a unit probe interval representation.

It is worth mentioning here that we use either Lemma 4.2.7 or Lemma 4.2.11 for the following proofs. We use Lemma 4.2.11 for the proofs that $F2$, $F3$, $F4$, $F4$, $F5$, $F6$, $F10$, $F11$ are forbidden subgraphs for unit probe interval graphs and Lemma 4.2.7 for proving that $F7$, $F8$, $F9$ are also forbidden subgraphs for unit probe interval graphs.

Lemma 4.4.1 *The 2-path F2 (5-fan) given in Figure 4.14 is not a unit probe interval graph.*

Proof: Let us assume that $F2$ as shown in Figure 4.14 is a UPIG. It can be observed from the figure that $u_2, u_3, u_4, u_5, u_6, u_7$ and $u_2, u_1, u_3, u_4, u_5, u_6$ are vertices of two 4-fans and both of them have center at u_2. The central radial vertex of the first 4-fan is u_5 and the central radial vertex of the second 4-fan is u_4. Hence by lemma 4.2.11 u_4 and u_5 must be non-probes. But $u_4 \leftrightarrow u_5$ and hence we get a contradiction. Thus $F2$ is not a unit probe interval graph. ■

Lemma 4.4.2 *The 2-path* $F3$ *given in Figure 4.15 is not a unit probe interval graph.*

Proof: Let us assume that $F3$ is a UPIG. We can see from Figure 4.15 that x_{-1}, x_1, x_8 and x_{10} are centers of 4-fans and x_{-2}, x_2, x_7 and x_{11} are central radial vertices

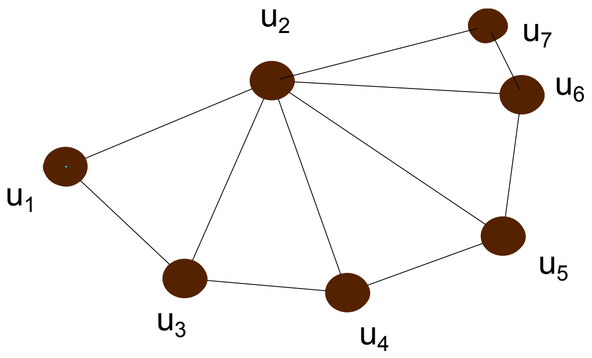


Figure 4.14: F2

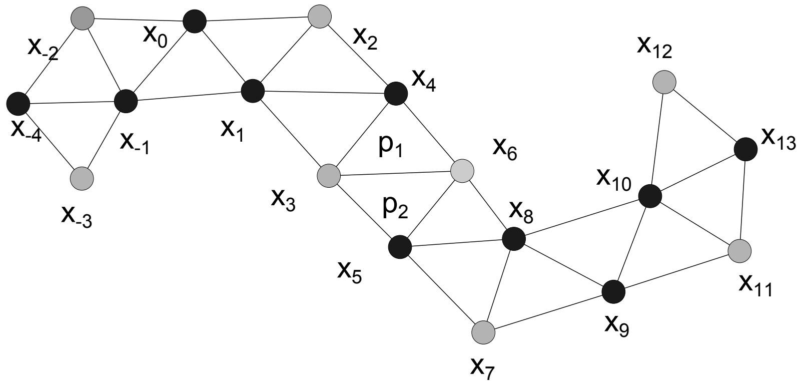


Figure 4.15: F3

of the four 4-fans. Also x_{-1} and x_1 are end radial vertices of the 4-fans with centers at x_1 and x_{-1} respectively. So x_1 and x_{-1} must be probes. Furthermore x_8 and x_{10} are also end radial vertices of the 4-fans with centers at x_{10} and x_8 respectively. So x_{10} and x_8 must be probes as well. Hence x_{-3}, x_3, x_6 and x_{12} the other end radial vertices of the 4-fans must be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction.

■

Lemma 4.4.3 *The 2-path F4 given in Figure 4.16 is not a unit probe interval graph.*

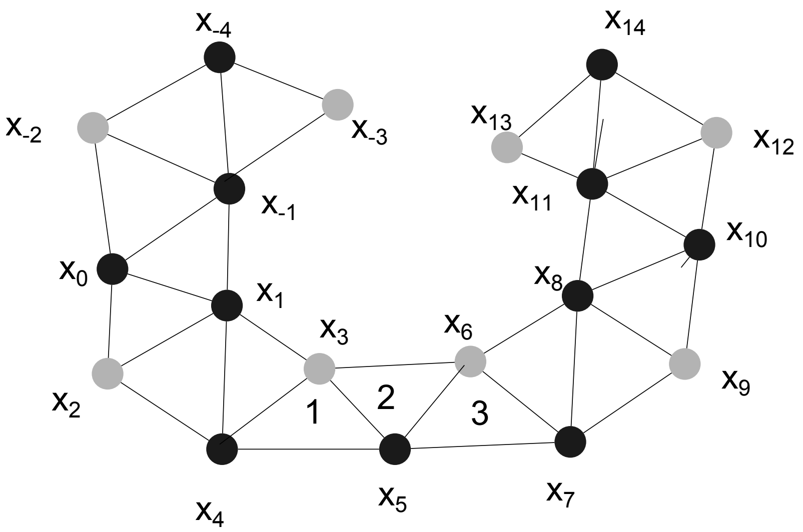


Figure 4.16: F4

Proof: Let us assume that $F4$ is a UPIG. We can see from Figure 4.16 that x_{-1}, x_1, x_8 and x_{11} are centers of 4-fans and x_{-2}, x_2, x_9 and x_{12} are central radial vertices of the four 4-fans. Also x_{-1} and x_1 are end radial vertices of the 4-fans with centers at x_1 and x_{-1} respectively. So they must be probes. Furthermore x_8 and x_{11} are also end radial vertices of the 4-fans with centers at x_{11} and x_8 respectively. So they must be probes too. Hence x_{-3}, x_3, x_6 and x_{13} the other end radial vertices of the 4-fans must be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction. ■

Lemma 4.4.4 *The 2-path F5 given in Figure 4.17 is not a unit probe interval graph.*

Proof: Let us assume that $F5$ is a UPIG. We can see from Figure 4.17 that x_{-1}, x_1, x_8 and x_{12} are centers of 4-fans and x_{-2}, x_2, x_9 and x_{13} are central radial vertices of the four 4-fans. Also x_{-1} and x_1 are end radial vertices of the 4-fans with centers at x_1 and x_{-1} respectively. So x_1 and x_{-1} must be probes. Furthermore x_8 and x_{12} are also end radial vertices of the 4-fans with centers at x_{12} and x_8 respectively. So x_{12}

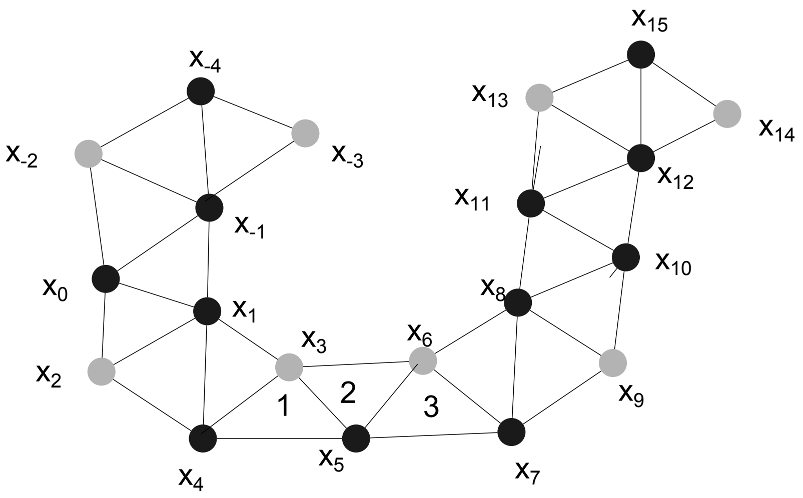


Figure 4.17: F5

and x_8 must be probes too. Hence x_{-3}, x_3, x_6 and x_{14} the other end radial vertices of the 4-fans must be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction. ■

Lemma 4.4.5 *The 2-path F6 given in Figure 4.18 is not a unit probe interval graph.*

Proof: Let us assume that $F6$ is a UPIG. We can see from Figure 4.18 x_{-2}, x_1, x_9 and x_{12} are centers of 4-fans and x_{-3}, x_2, x_8 and x_{13} are central radial vertices of the four 4-fans. Hence x_{-1} and x_{11} cannot be non-probes which are also the end radial vertices for the second and the third 4-fan. So x_3 and x_6 the other end radial vertices of the 4-fans must be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction. ■

Lemma 4.4.6 *The 2-path F7 given in Figure 4.19 is not a unit probe interval graph.*

Proof: Let us consider a 2-tree probe interval graph $F7$ as in Figure 4.19. Assume that $F7$ has a unit probe interval representation. Note that u_4 is the center of an induced $K_{1,3}$ formed by u_4, u_2, u_3 and u_6. So by Lemma 4.2.7 u_4 must be a probe. Note that u_6 is also a center of an induced $K_{1,3}$ formed by u_6, u_4, u_7 and u_8. So u_6 must be a probe and at least two of u_4, u_7 and u_8 must be non-probes. Since

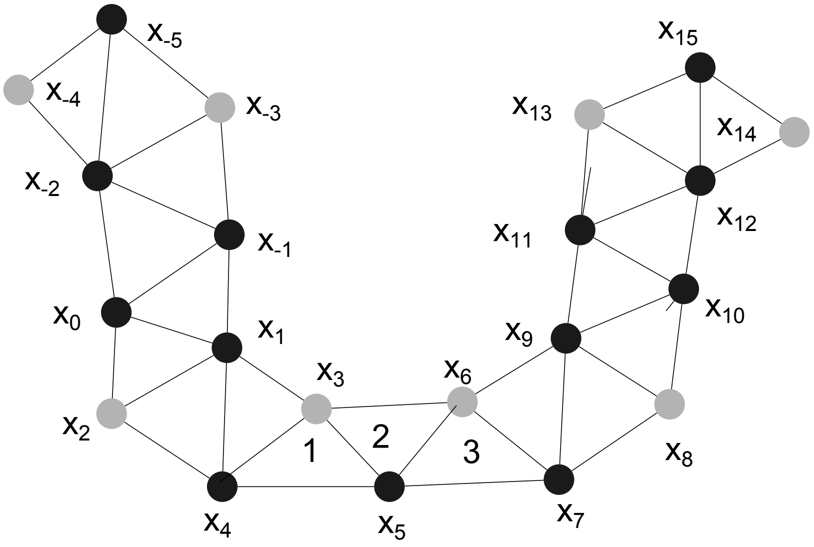


Figure 4.18: F6

u_4 is a probe, u_7 and u_8 must be non-probes. But u_8 is the center of another induced $K_{1,3}$ with leaves u_6, u_{10} and u_{12} and so u_8 must be a probe which is a contradiction.

■

Lemma 4.4.7 *The 2-path F8 given in Figure 4.20 is not a unit probe interval graph.*

Proof: Let us consider a 2-tree probe interval graph $F8$ which is a merge of three 4-fans as in Figure 4.20 with centers at u_4, u_{11} and u_6. Let us assume that F8 has a unit probe interval representation. The vertex u_4 is the center of an induced $K_{1,3}$ formed by u_4, u_2, u_3 and u_6 so u_4 must be a probe. Note that u_6 is also a center of an induced $K_{1,3}$ with leaves u_4, u_7 and u_8; so u_6 must be a probe and at least two of u_4, u_7 and u_8 must be non-probes. Since u_4 is already a probe, then u_7 and u_8 must be non-probes. Also u_9 and u_{10} are adjacent to u_8, so u_9 and u_{10} must be probes. Now, u_{11}, u_9, u_{10} and u_{13} form an induced $K_{1,3}$ with center u_{11}. So at least two vertices out of u_9, u_{10} and u_{13} must be non-probes. But u_9 and $, u_{10}$ are probes, so u_{13} is the only vertex which is a non-probe and hence we get a contradiction.

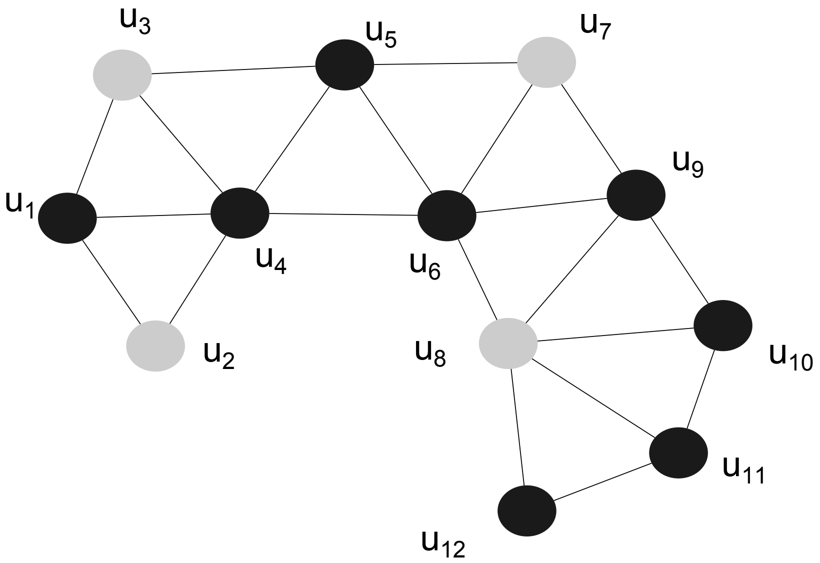


Figure 4.19: F7

■

Lemma 4.4.8 *The 2-path F9 given in Figure 4.21 is not a unit probe interval graph.*

Proof: Let us consider a 2-tree probe interval graph $F9$ which is a merge of three 4-fans $G1, G2$ and $G3$ with centers at x_{-2}, x_1 and x_6 as in Figure 4.21. let us assume that $F9$ has a unit probe interval representation. x_{-2} is the center of an induced $K_{1,3}$ formed by x_{-2}, x_{-4}, x_{-3} and x_0. So x_{-2} must be a probe and either x_{-3}

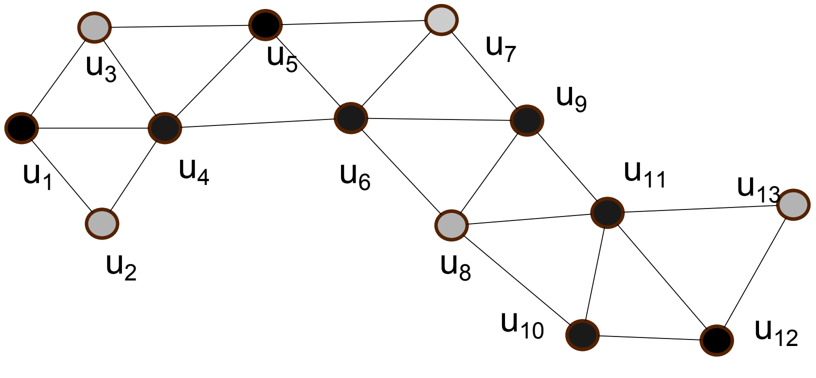


Figure 4.20: F8

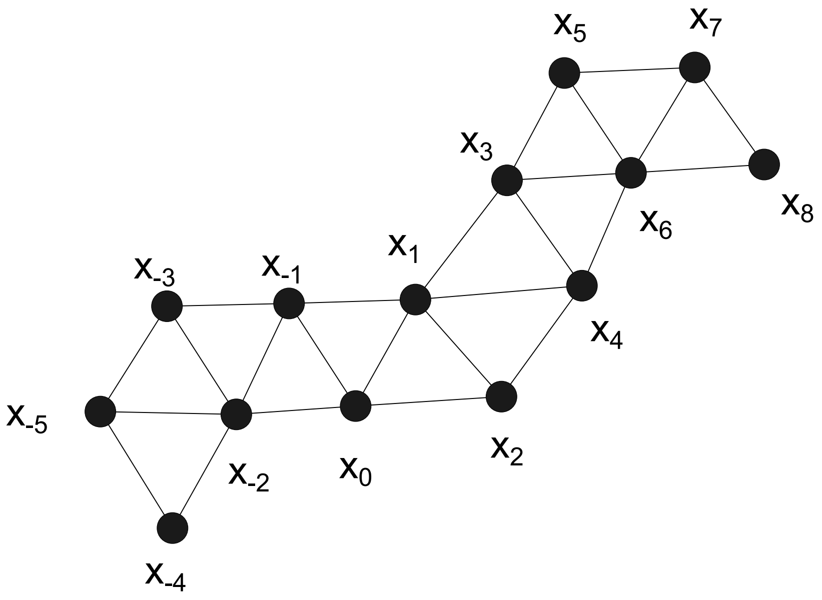


Figure 4.21: F9

and x_0 or x_{-3} and x_{-4} are non-probes. Again x_1 is the center of an induced $K_{1,3}$ formed by x_1, x_{-1}, x_2 and x_3. So x_1 must be a probe and either x_{-1} and x_2 or x_2 and x_3 are non-probes.

Case 1: If x_{-3} and x_0 are non-probes, then x_{-1} must be a probe. So x_2 and x_3 must be non-probes by Lemma 4.2.7 and 4.2.11 which implies that two adjacent vertices x_0 and x_2 are both non-probes which is a contradiction. Hence x_{-3} and x_0 cannot be both non-probes.

Case 2: If x_{-3} and x_{-4} are non-probes then x_{-1} must be a probe. So x_2 and x_3 must be non-probes by Lemmas 4.2.7 and 4.2.11 which implies that x_4 and x_5 are probes. From $G3$ whose center is at x_6 we can see that x_6, x_4, x_5 and x_8 form an induced $K_{1,3}$ with center x_6, and so at least two of x_4, x_5 or x_8 must be non-probes. But x_4 and x_5 are already probes and so we get a contradiction. Hence It is not possible to construct a unit probe interval representation of $F9$.

■

Lemma 4.4.9 *The 2-path $F10$ given in Figure 4.22 is not a unit probe interval graph.*

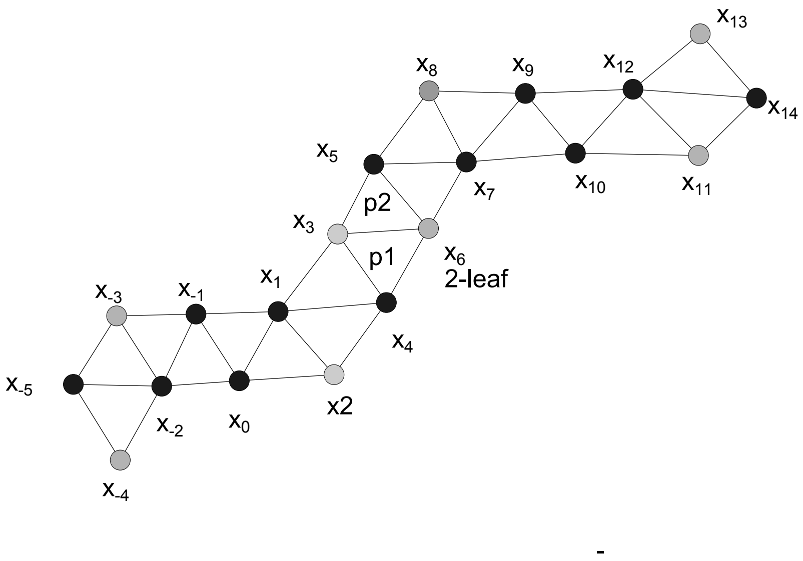


Figure 4.22: F10

Proof: Let us assume that $F10$ is a UPIG. We can see from the Figure 4.22 that x_{-2}, x_1, x_7 and x_{12} are centers of 4-fans. It can also be observed that x_{-3}, x_2, x_8 and x_{11} are central radial vertices of the four 4-fans, and thus must be non-probes. Hence x_{-1} x_0, x_9 and x_{10} must be probes. Each of these vertices is also one of the end radial vertices of the 4-fans. So the other end radial vertices x_{-4}, x_3, x_6 and x_{13} of the 4-fans must be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction. ■

Lemma 4.4.10 *The 2-path F11 given in Figure 4.23 is not a unit probe interval graph.*

Proof: Let us assume that $F11$ is a UPIG. We can see from the Figure 4.23 that x_{-2}, x_1, x_7 and x_{10} are centers of 4-fans where x_{-3}, x_2, x_8 and x_{12} are central radial vertices of the four 4-fans. So x_{-3}, x_2, x_8 and x_{12} must be non-probes. Hence x_{-1} and x_0 must be probes and they are the end radial vertices of 4-fans with centers at x_1 and x_{-2}. Furthermore x_7 and x_{10} are also end radial vertices of the 4-fans with centers at x_{10} and x_7 respectively. Since x_{10} and x_7 are also centers, they must be probes. Hence the other end radial vertices, x_{-4}, x_3, x_6 and x_{11} of the 4-fans, must

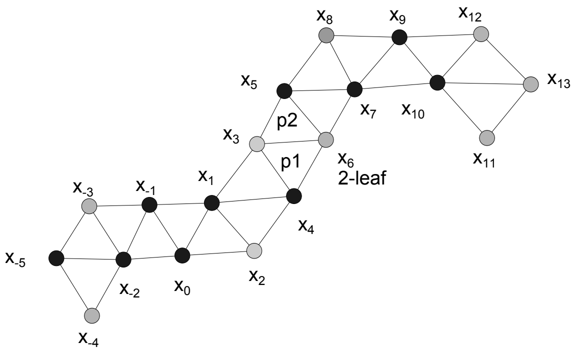


Figure 4.23: F11

be non-probes. But $x_3 \leftrightarrow x_6$ and so we get a contradiction. ■

4.5 2-path unit probe interval graph characterization

In the following section we will give a characterization of 2-paths that are unit probe interval graphs. The 2-path characterization turned out to be quite difficult and, the complete list of forbidden subgraphs for this characterization includes 10 subgraphs. In the beginning of the section we will give several definitions which have been used to proceed through the proofs. We will also introduce some additional definitions as we move towards the final goal. While constructing a 2-path we start with a 2-complete graph, K_2 and call it G. Next we keep adding new vertices adjacent to a 2-complete subgraph of G such that the new vertex is adjacent either to the two most recently added vertices or not. If it is adjacent to the most recently added vertices we get a 2-path with no 4-fan which we will call a straight 2-path (we will formally define a straight 2-path later). If not, then the new vertex, after addition, either forms an $F1$ which is not allowed since it is not a 2-path or it creates a 4-fan. So at every stage we have exactly three places where the new vertex can be added provided no induced $F2$......$F11$ is formed and as a result we either get straight 2-paths, $F1$s or merge of 4-fans. Hence it can be concluded from the discussion so far that with addition of a vertex to an existing 2-path either a 4-fan is created or not. Thus the structures of a 2-path avoiding the forbidden subgraphs from $F2, \ldots\ldots F11$, can either be a straight 2-path or merge of 4-fans. At first we figure out the possible ways 4-fans can be merged to each other.

Definition 4.5.1 *Edge-consecutive 4-fans: Let A_i, A_{i+1} be two 4-fans. The 4-fans A_i, A_{i+1} will be called edge-consecutive if A_i shares at least one edge with A_{i+1}, i=1,....(n-1).*

Definition 4.5.2 *Vertex-consecutive 4-fans: Let A_i, A_{i+1} be two 4-fans. The 4-fans A_i, A_{i+1} will be called vertex-consecutive if A_i shares exactly one vertex with A_{i+1}.*

Now we will prove some useful lemmas using the above definitions to find the possible shapes of a 2-path. First we will deal with 2-paths which are just merge of 4-fans. As stated earlier 4-fans in a 2-path can be either edge-consecutive or vertex-consecutive. First we determine the possible structures of edge-consecutive 4-fans.

Lemma 4.5.3 *Two edge-consecutive 4-fans without an F2 (5-fan) cannot share more than two K_3s.*

Proof: Let $t_1 - t_2 - t_3 - t_4$ $(e_0, t_1, \ldots..t_4, e_4)$ be a 4-fan A. let us add another 4-fan B to this such that t_2, t_3 and t_4 also belongs to B. In other words t_2, t_3 and t_4 are the shared K_3. The only way we get a 4-fan from t_2, t_3 and t_4 is by making a new vertex adjacent to e_4 as shown in picture (1) of the Figure 4.25. It is easy to see that we get an F2 here which is a contradiction. Hence two 4-fans cannot share more than two K_3s. ■

Lemma 4.5.4 *Two edge-consecutive 4-fans without an F2 (5-fan) can share two K_3s in exactly one way as A_6^4 in the Figure 4.24 .*

Proof: Let A1 and A2 be two edge-consecutive 4-fans without an F2 (5-fan) such that they share exactly two K_3s. Let us call this graph G. Since A1 and A2

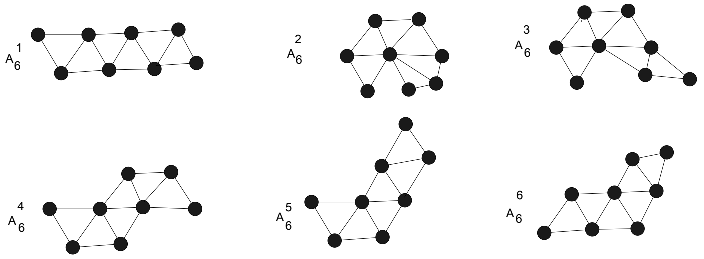


Figure 4.24: Non-isomorphic 2-paths of length 6

share exactly 2 K_3s, then G must contain exactly six K_3s. In other words G must be a 2-path of length 6. It has been proved in Pržulj and Corneil that there are six non-isomorphic 2-paths of length 6 ($A_6^1, A_6^2, A_6^3, A_6^4, A_6^5, A_6^6$) as shown in Figure 4.24. It can be seen in the figure that A_6^1 contains no 4-fan, and that A_6^5, A_6^6 have exactly one induced 4-fan. Further A_6^2, A_6^3 have an induced F2 in them. Thus $A_6^1, A_6^2, A_6^3, A_6^5, A_6^6$ cannot be G. Hence G must be A_6^4 ■

The following lemma will determine the possible structures any two edge-consecutive 4-fans without an F2 can have.

Lemma 4.5.5 *If G1 and G2 are two edge-consecutive 4-fans that form a 2-path with no* $F2$*, then G1 and G2 together are one of the graphs in Figure 4.27.*

Proof: Let $t_1 - t_2 - t_3 - t_4$ $(e_0, t_1, ..., t_4, e_4)$ be a 4-fan. Call it G_1. Let G_2 be another 4-fan. We can make these two 4-fans edge-consecutive without forming an induced $F2$ in the following ways (keeping in mind that they cannot share more than two 3-cliques):

Case1: G_1 and G_2 share exactly two K_3s:

This case has been dealt with in the previous lemma. Its construction is illustrated in the pictures 2(a), 2(b), 2(c) of Figure 4.25. 2(b) and 2(c) contain an $F1$ (which is not a 2-path)and an $F2$ and so we do not consider them.

Case2: G_1 and G_2 share exactly one K_3:

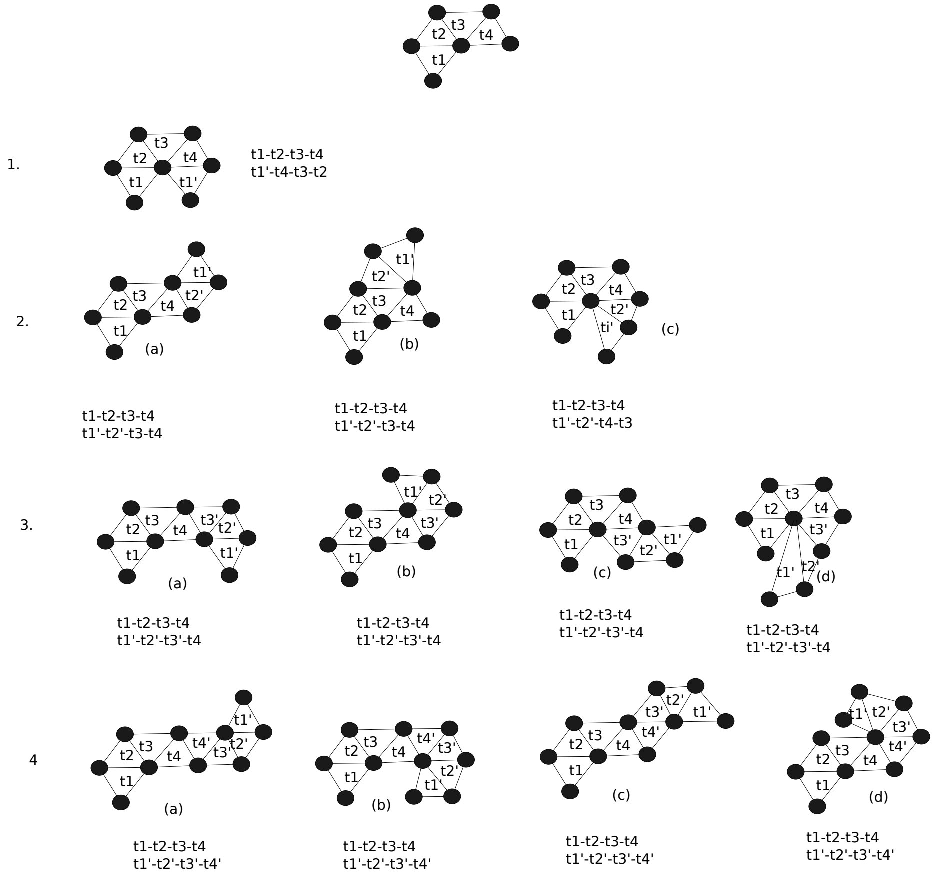


Figure 4.25: Construction that shows possible merge of two-4-fans

In this case we will merge G_1=$t_1 - t_2 - t_3 - t_4$ with G_2 such that the 3-clique t_4 becomes the first 3-clique of G_2. Since t_4 has two available edges to which the vertex from G_2 can be adjacent, we get exactly 2 ways to add it. One of the ways form an $F2$ as shown in picture 3(c) of Figure 4.25 and so we do not consider it any more. Thus we construct $t_4 - t_3^{'}$ of G_2 where $t_3^{'}$ is the second 3-clique of G_2. Now at this stage we are just left with 2 options for the next vertex from G_2 which again in turn has just one option in each case for the final vertex of G_2. Thus we get the structures 3(a) and 3(b) of the Figure 4.25. 3(b) also has an induced $F2$ and so we no longer consider it.

Case3: G_1 and G_2 share exactly one edge:

In this case we will merge G_1=$t_1 - t_2 - t_3 - t_4$ with G_2 such that the edge e_4 becomes the first edge of G_2. At this stage we have exactly one way to add a new vertex. After this we have exactly two ways to add the new vertex. One leads to 4(a), 4(b) and the other leads to 4(c), 4 (d) of Figure 4.25. This construction is illustrated in Figure 4.26. Here 4(b) and 4(d) have an induced F2 and so we do not consider them. 4(c) is an extension of 2(a).

So our final structures are given by 2(a), 3(a) and 4(a) as illustrated in Figure 4.25.

■

Note: 2(a) of Figure 4.25 is formed if two edge-consecutive 4-fans share exactly two K_3, 3(a) of Figure 4.25 is formed if two edge-consecutive 4-fans share exactly one K_3 and 4(a) of Figure 4.25 is formed if two edge-consecutive 4-fans share exactly one edge.

Lemma 4.5.6 *If a graph is a merge of three graphs G_1, G_2, G_3 which are three edge-consecutive 4-fans that form a 2-path with no $F2, F7, F8, F9$ then it is one of the following graphs in Figure 4.31.*

Proof: We consider 3 cases here:

Case1: G_1 and G_2 share exactly two K_3s and G3 shares either two K_3s, one K_3 or one edge with G_2. Since 2(a), 3(a) and 4(a) as illustrated in Figure 4.25 are the possible structures here. We start with G_1G_2 as the graph 2(a) in Figure 4.25. So now we have to look at the case where G_2 can be edge consecutive with G_3 with no $F2$. We know (by Lemma 4.5.5) that it can be done in exactly 3 ways and hence we get 3 structures as shown in Figure 4.28.

Case2: G_1 and G_2 share exactly one K_3 and G_3 shares either two K_3s, one K_3 or one edge with G_2. Since 2(a), 3(a) and 4(a) as illustrated in Figure 4.25 are the possible structures here. We start with G_1G_2 as the graph 3a in figure 4.25. So now we have to look at the case where $G2$ can be edge consecutive with G_3 with no $F2$. We know (by lemma 4.5.5) that it can be done in exactly 3 ways and hence we get 3 structures as shown in the Figure 4.29.

Case3: G_1 and G_2 share exactly one edge and G3 shares either two K_3s, one K_3 or one edge with G_2. Since 2(a), 3(a) and 4(a) as illustrated in Figure 4.25 are the possible structures here. We start with G_1G_2 as the graph 4(a) in figure 4.25. As in the previous cases, now we look at the case that G_2 can be edge consecutive with G_3 with no $F2$. Again we know (by Lemma 4.5.5) that it can be done in exactly 3 ways and hence we get 3 structures as shown in Figure 4.30.

Since (3),(2) from Figure 4.28 are isomorphic to (7) from Figure 4.30, and (4) from Figure 4.29 respectively and (6) from Figure 4.29 is isomorphic to (8) from figure 4.30, we get six non-isomorphic structures which are illustrated in figure 4.31 three of which are forbidden subgraphs $F7$, $F8$, $F9$.

■

We now look at the vertex-consecutive case. We already know that edge-consecutive 4-fans are capable of assuming exactly 3 structures. The following two lemmas will prove that vertex-consecutive 4-fans can assume exactly one structure

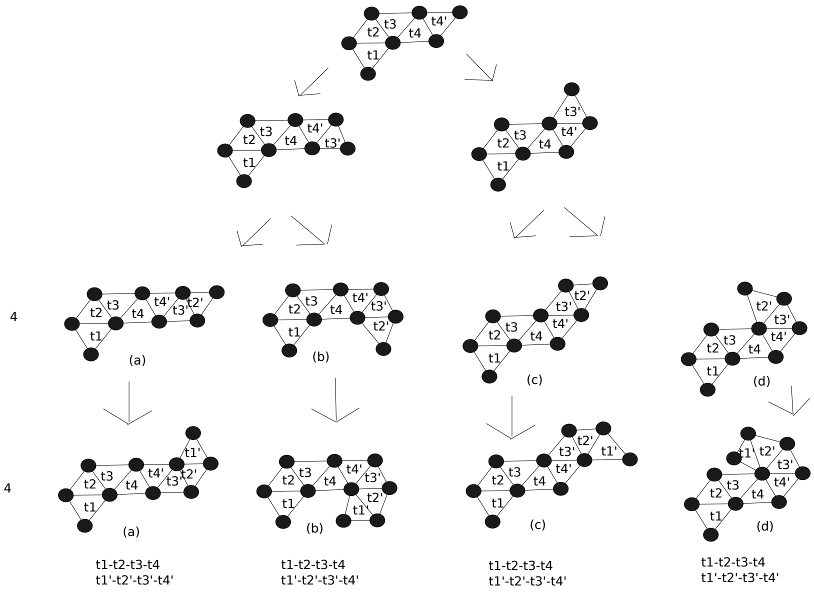


Figure 4.26: Construction of two edge-consecutive 4-fans

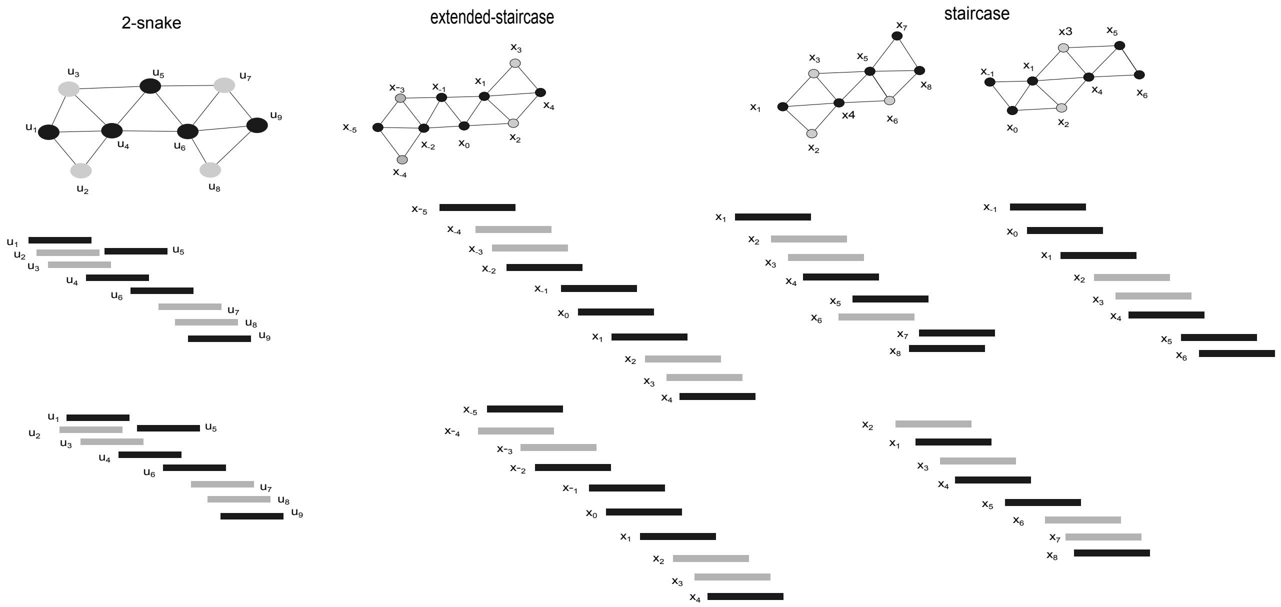


Figure 4.27: All possible two edge-consecutive 4-fans and their representations

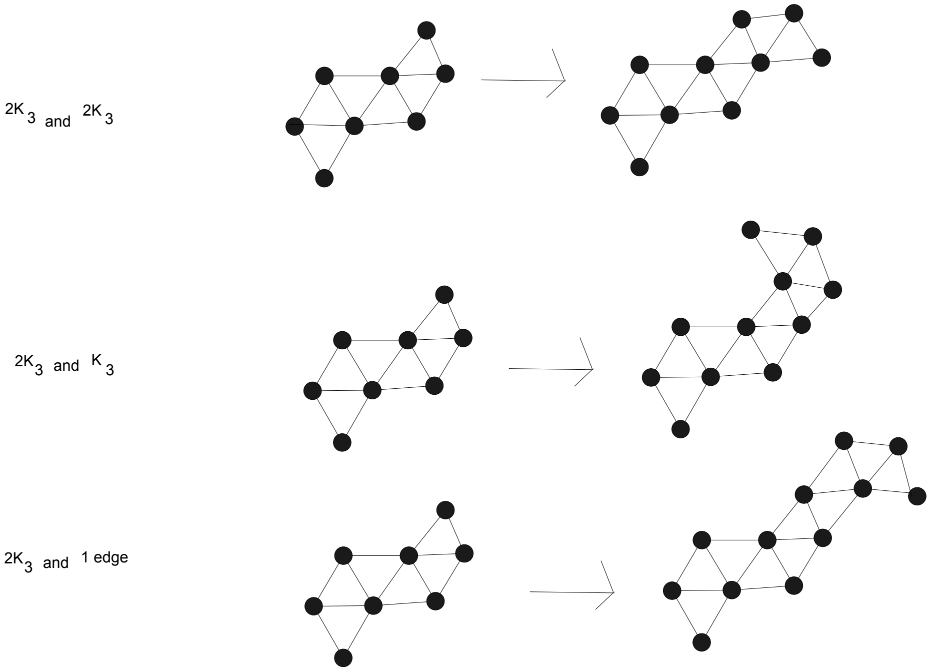


Figure 4.28: Formation of three edge-consecutive-4-fans

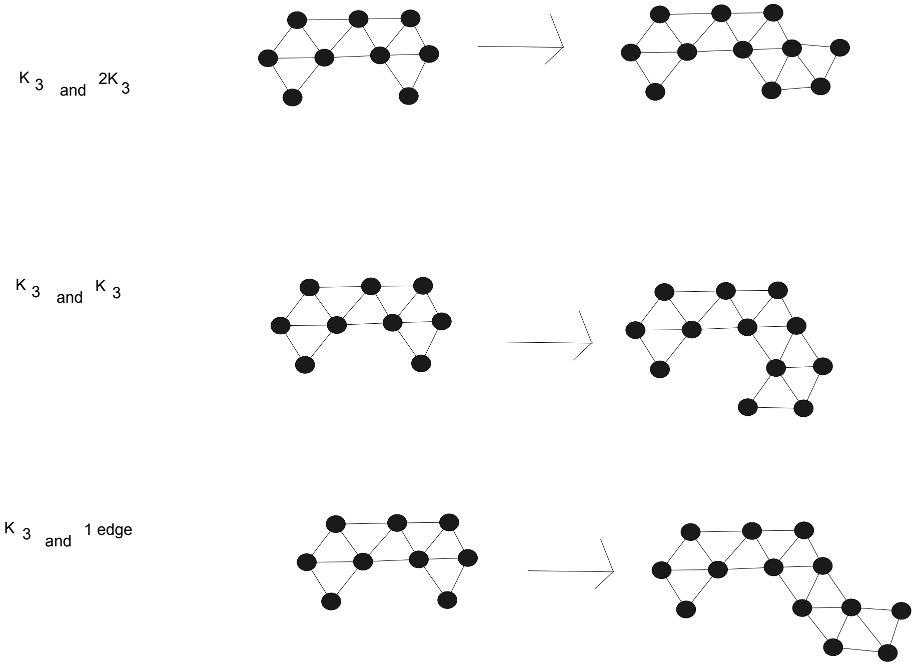


Figure 4.29: Formation of three edge-consecutive-4-fans

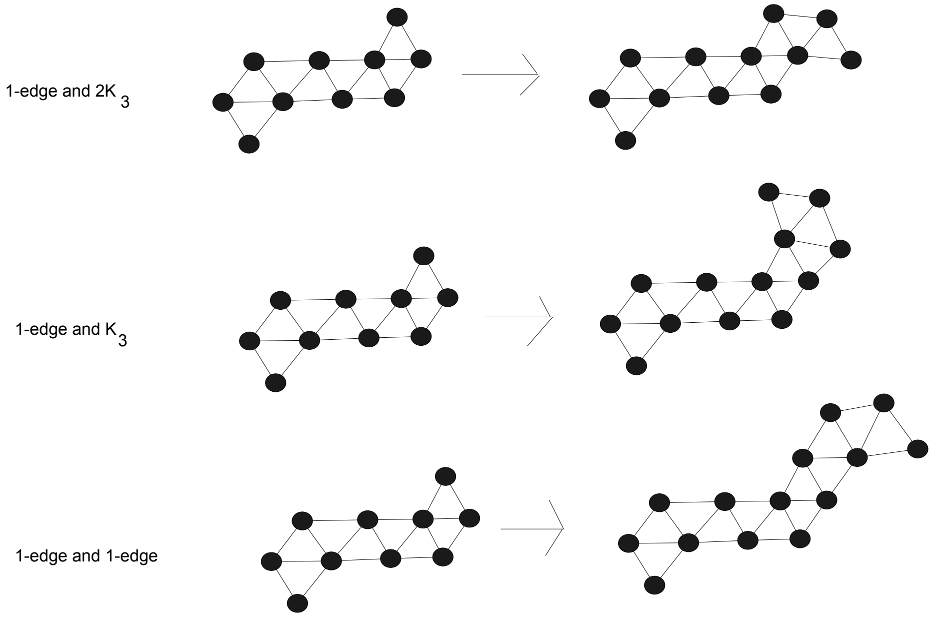


Figure 4.30: Formation of three edge-consecutive-4-fans

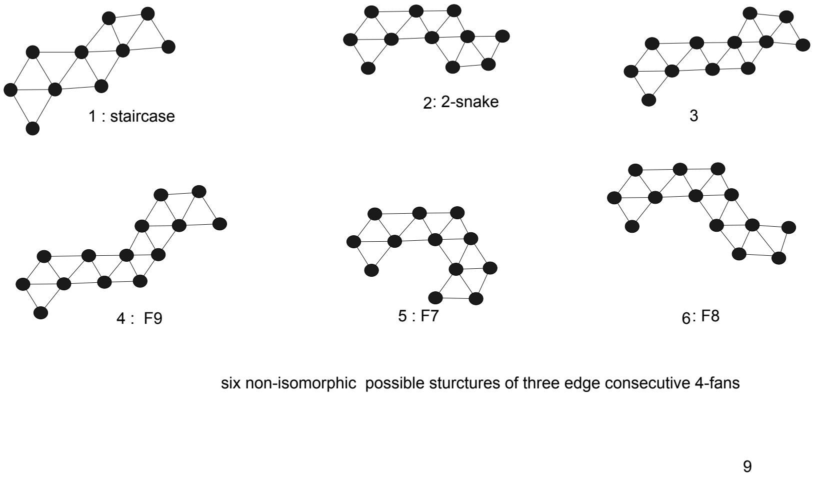

1 : staircase
2: 2-snake
3
4 : F9
5 : F7
6: F8
six non-isomorphic possible sturctures of three edge consecutive 4-fans
9

Figure 4.31: three edge-consecutive-4-fans

which is also illustrated below.

Lemma 4.5.7 *If G1 and G2 are two vertex-consecutive 4-fans that form a 2-path with no F2 then it is the graph in Figure 4.33.*

Proof: Let G be the merge of G_1 and G_2. Since G forms a 2-path, it cannot have $F1$ as an induced subgraph. It can be observed that G_1 and G_2 cannot share any non-end radial vertex since in that case three K_3s from one and one K_3 from its complement would form an $F1$. Hence G_1 and G_2 can share only the end vertices. Hence if $G_1 = u_1 - u_2 - u_3 - u_4 - u_5 - u_6$ with center at u_4 and $G_2 = u_6 - u_7 - u_8 - u_9 - u_{10} - u_{11}$ with center at u_8 then u_6 is the shared vertex. Thus we get the possible structure of G which is illustrated in Figure 4.33.
A construction of G from G_1 is also illustrated in figure 4.32. ■

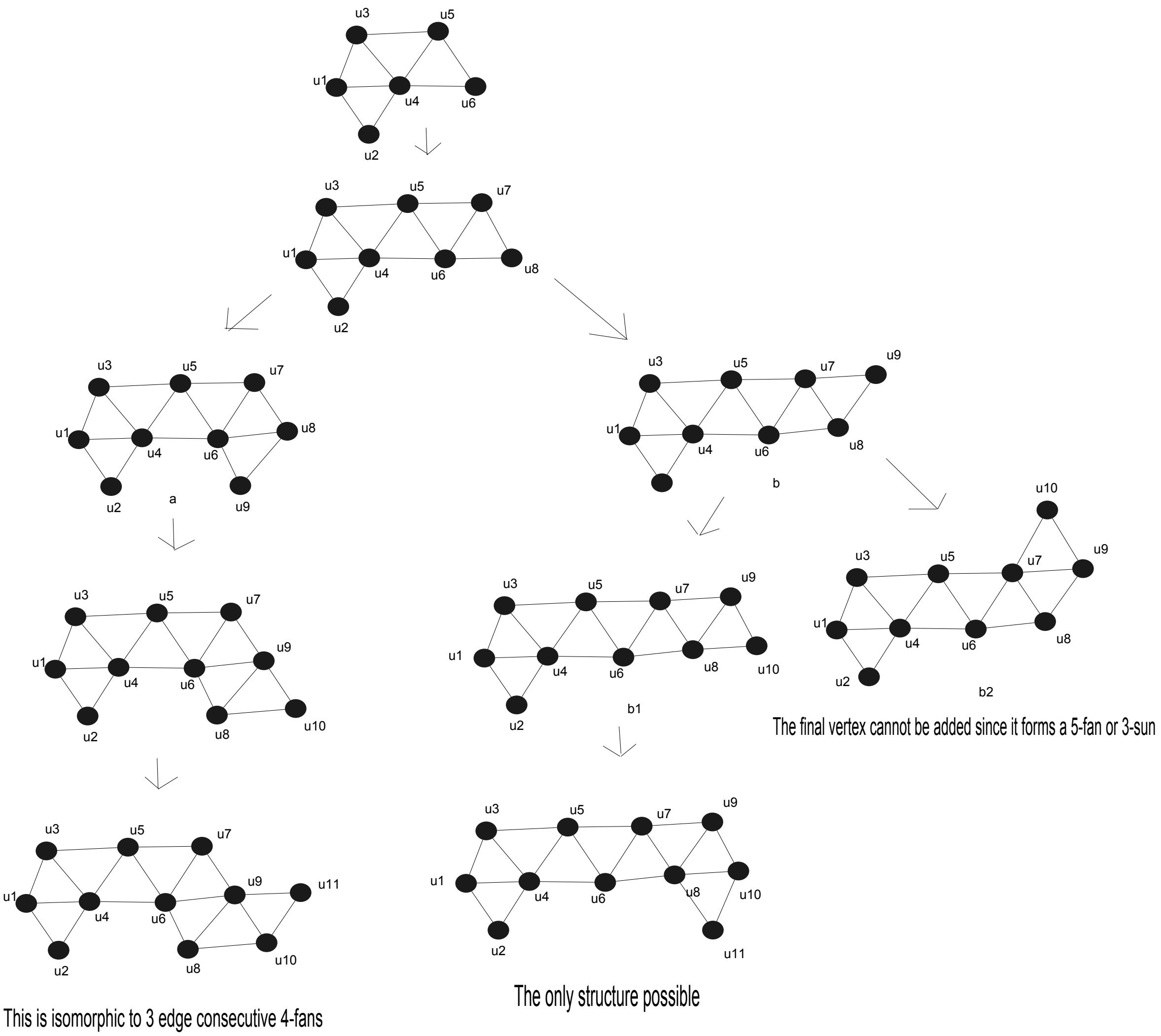
Lemma 4.5.8 *If G1, G2, G3 are vertex-consecutive 4-fans that form a 2-path with no F2 then it is the graph in Figure 4.34.*

Proof: G_1 and G_2 share exactly one vertex and G_3 shares one vertex with G_2. So now we have to look at the ways G_2 can be vertex-consecutive with G_3 with no $F2$. From the previous lemma we know that it can be done in exactly 1 way and hence we get only one structure as shown in Figure 4.34.

■

Now we define some possible structures of a 2-path which have nice unit probe interval representations. The representations are also given below. We will also use the operation + to denote merge between 4-fans. The sign plus is used mostly in Figures for convenience.

Definition 4.5.9 *Straight 2-path : A straight 2-path is a 2-path with no 4-fans in the structure. An example is illustrated in Figure 4.37.*



The final vertex cannot be added since it forms a 5-fan or 3-sun

This is isomorphic to 3 edge consecutive 4-fans

The only structure possible

Figure 4.32: Formation of two vertex-consecutive-4-fans

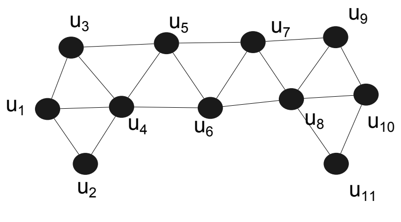


Figure 4.33: The only structure possible for two vertex-consecutive-4-fans

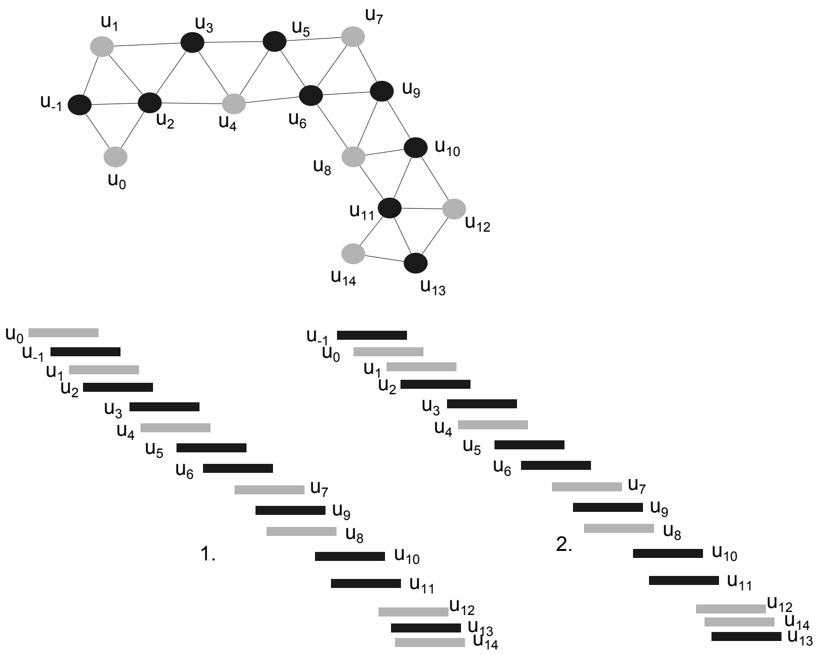


Figure 4.34: Representations of three vertex-consecutive-4-fans

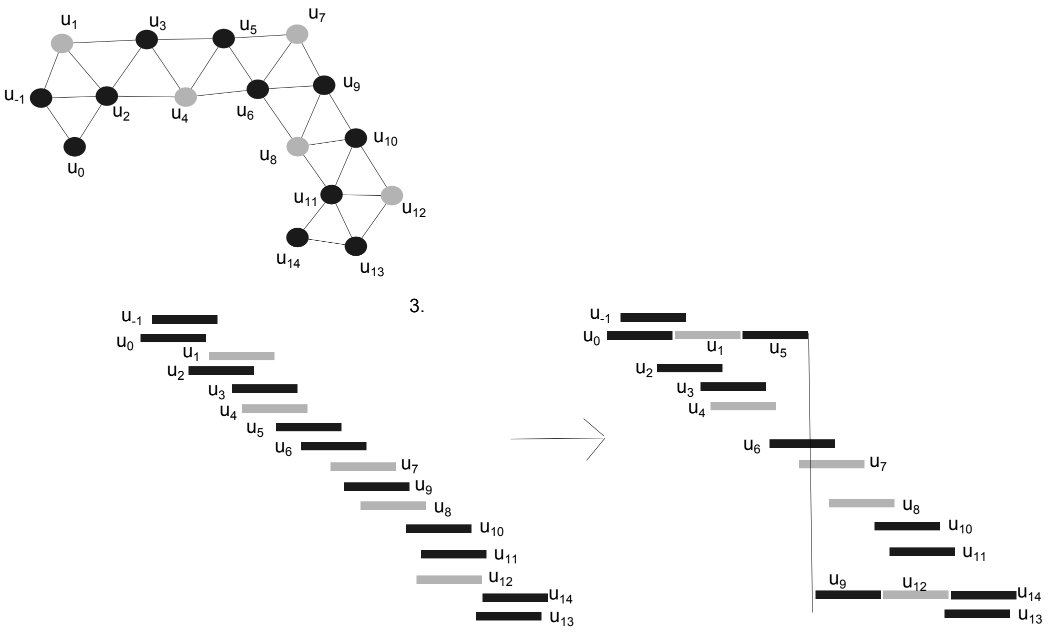


Figure 4.35: Another representation of three vertex-consecutive-4-fans

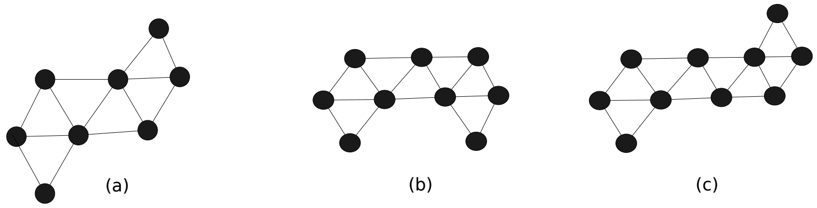


Figure 4.36: Three possible structures of two edge-consecutive 4-fans

Definition 4.5.10 *Staircase : A 2-path is a staircase if it is a merge of 4-fans where the consecutive 4-fans share exactly two 3-cliques. An example is illustrated in Figure 4.36a.*

Definition 4.5.11 *2-snake : A 2-path is a 2-snake if it is a merge of 4-fans where the consecutive 4-fans share exactly two or one 3-cliques and the centers of all the 4-fans form a path. An example is illustrated in Figure 4.36b.*

Definition 4.5.12 *Extended staircase :A 2-path is an extended staircase if it is a merge of 4-fans where the consecutive 4-fans share exactly one edge. This is illustrated in Figure 4.36c.*

The following lemma is a very useful result and helps a lot in later proofs where we give the unit probe interval representations of the 2-paths.

Lemma 4.5.13 *The 2-leaves on both ends of a 2-snake and an extended staircase must always be non-probes.*

Proof: Let us first consider a 2-snake as shown in Figure 4.27 (1). Here u_4 and u_6 are the centers of two 4-fans, so they must be probes. Again u_4 and u_6 are also one of the end radial vertices of the 4-fans with center u_6 and u_4. Hence the other end radial vertices of the 4-fans u_2 and u_8 which are also the end 2-leaves of the 2-snake must be non-probes.

Let us now consider an extended staircase as shown in Figure 4.27 (2). Here x_{-2}

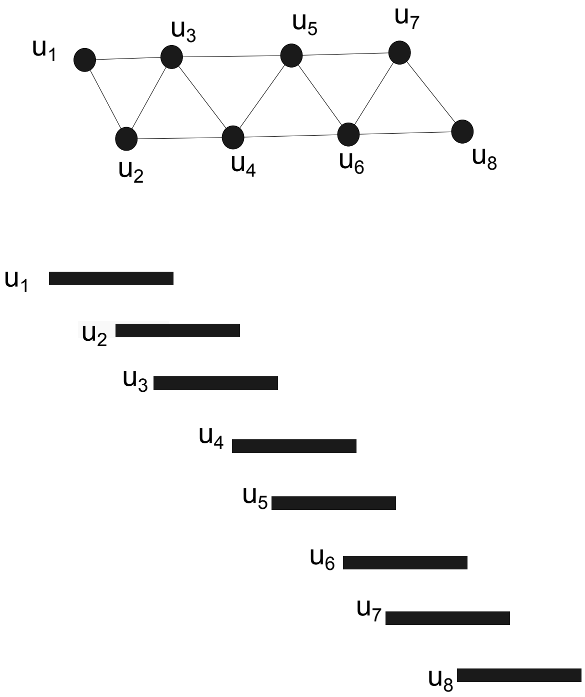


Figure 4.37: Straight 2-path

and x_1 are the centers of two 4-fans. So they must be probes. The central radial vertices x_{-3} and x_2 of the two 4-fans must be non-probes. So their adjacent vertices x_{-1} and x_0 must be probes. But x_{-1} and x_0 are also one of the end radial vertices of the 4-fans with centers at x_1 and x_{-2} respectively. Hence the other end radial vertices , x_{-4} and x_3, which are also the end 2-leaves of the extended staircase must be non-probes. ■

Lemma 4.5.14 *A straight 2-path is a unit probe interval graph.*

Proof: It can be easily seen that a straight 2-path does not have any induced $K_{1,3}$. Hence it is a unit interval graph. Therefore it is a unit probe interval graph. ■

Lemma 4.5.15 *A staircase is a unit probe interval graph.*

Proof:

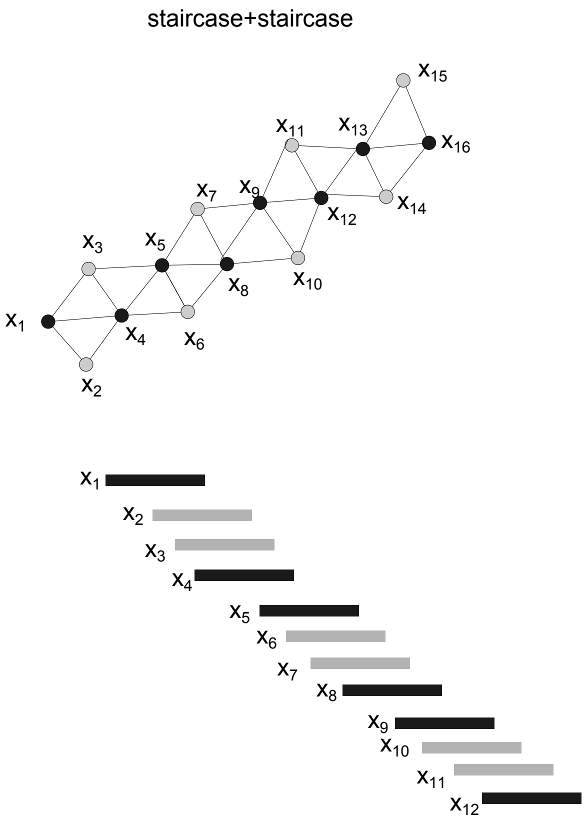


Figure 4.38: Staircase

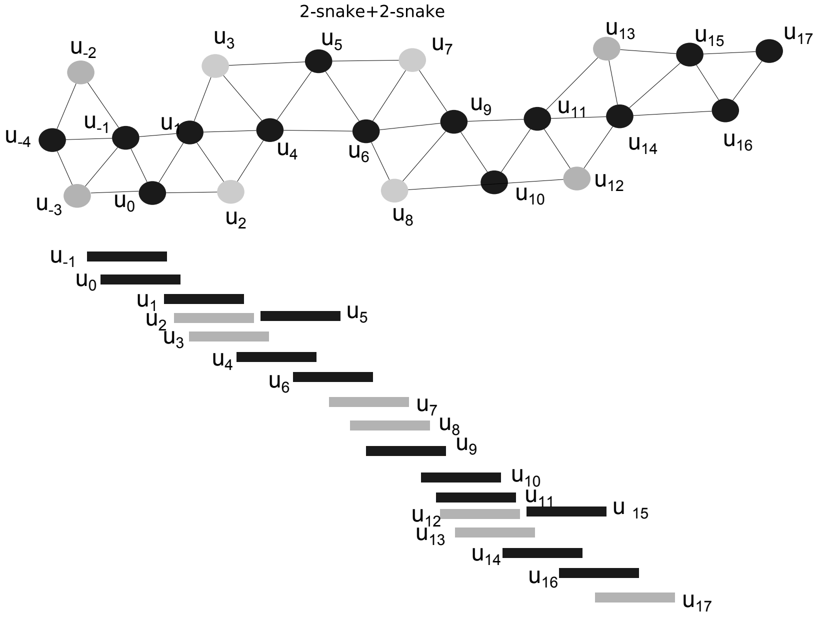
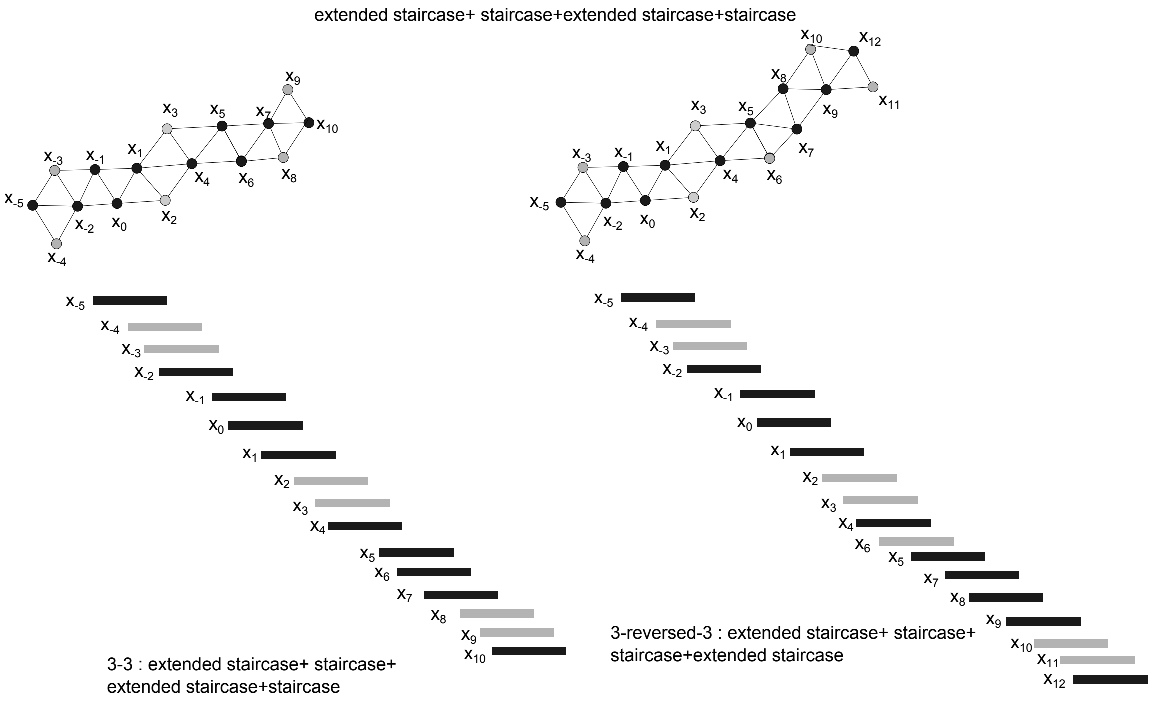


Figure 4.39: 2-snake



3-3 : extended staircase+ staircase+ extended staircase+staircase

3-reversed-3 : extended staircase+ staircase+ staircase+extended staircase

Figure 4.40: Extended-staircase+staircase

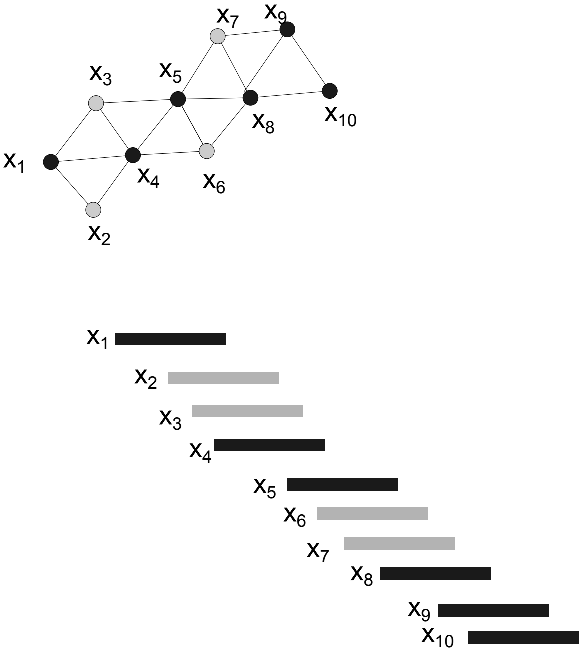


Figure 4.41: Staircase representation

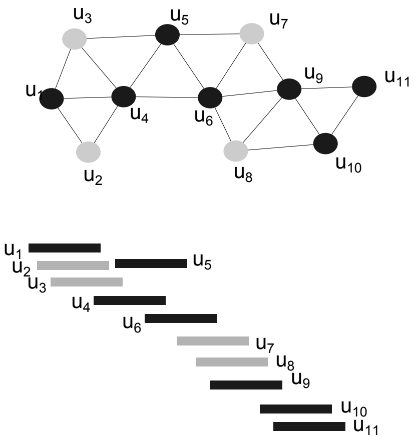


Figure 4.42: 2-snake representation

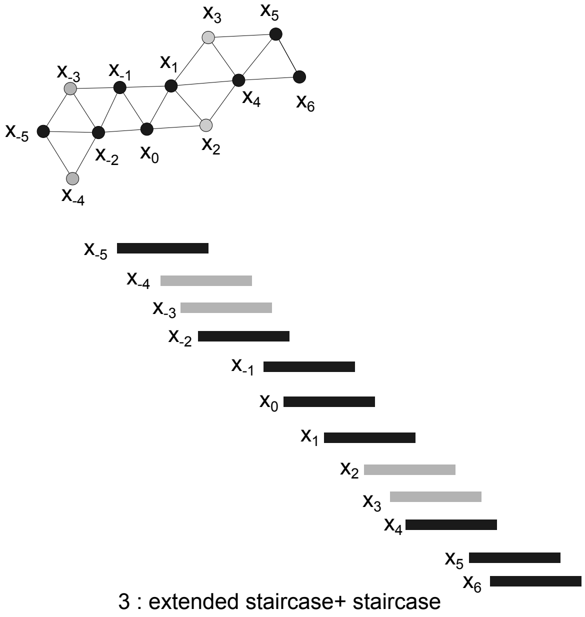


Figure 4.43: extended-staircase+staircase representation

Let G be a staircase as shown in Figure 4.38.

Let $x_2, x_3, x_6, x_7, x_{10}, x_{11}, x_{14}$ and x_{15} be non-probes. Thus the non-probes are vertices x_i, x_{i+1} where i starts at $i = 2$ and i is being updated by i:=i+3 after getting incremented by 1 in every iteration.

It is easy to see from the Figure 4.38 that if we start counting the 4-fans and the edges from the bottom then x_1x_4, x_5x_8, x_9x_{12} .. are the first interior edges of the first 4-fan, third 4-fan, fifth 4-fan... respectively. We will draw the intervals of these vertices such that:

$I_{x_1} \cap I_{x_4} = \epsilon$ such that the right end of x_1 overlaps with the left end of x_4.

$I_{x_5} \cap I_{x_8} = \epsilon$ such that the right end of x_5 overlaps with the left end of x_8.

$I_{x_9} \cap I_{x_{12}} = \epsilon$ such that the right end of x_9 overlaps with the left end of x_{12}.

Thus $I_{x_i} \cap I_{x_{i+3}} = \epsilon$ such that the right end of x_i overlaps with the left end of x_{i+3}. Starting at $i = 1$, the value of i gets updated for every iteration by $i = i + 4$. let us refer to the fan vertices from x_i to x_{i+3} for $i = 1, 5, 9, ...$ as vertices in boxes. Thus box_k, $k = 1, 2, 3, ...$ correspond to vertices $x_i - x_{i+3}$ for $i = 1, 5, 9, ...$ respectively.

$box_k = \{x_{1+4(k-1)}, ..., x_{4+4(k-1)}\}$

for box_1, let $r(I_{x_1}) < r(I_{x_2}) < r(I_{x_3}) < r(I_{x_4})$

for box_2, let $r(I_{x_5}) < r(I_{x_6}) < r(I_{x_7}) < r(I_{x_8})$

for box_3, let $r(I_{x_9}) < r(I_{x_{10}}) < r(I_{x_{11}}) < r(I_{x_{12}})$

Thus for any box_k, let $r(I_{x_j}) < r(I_{x_{j+1}}) < r(I_{x_{j+2}}) < r(I_{x_{j+3}})$

where x_j is the first vertex of the box.

For the transition from one box to another we do the following:

$r(I_{x_4}) < r(I_{x_5})$ such that $r(I_{x_2}) < l(I_{x_5})$ and $r(I_{x_3}) \cap l(I_{x_5}) = \epsilon$

$r(I_{x_8}) < r(I_{x_9})$ such that $r(I_{x_6}) < l(I_{x_9})$ and $r(I_{x_7}) \cap l(I_{x_9}) = \epsilon$

$r(I_{x_{12}}) < r(I_{x_{13}})$ such that $r(I_{x_{10}}) < l(I_{x_{13}})$ and $r(I_{x_{11}}) \cap l(I_{x_{13}}) = \epsilon$

Thus for each transition step $r(I_{x_i}) < r(I_{x_{i+1}})$ such that $r(I_{x_{i-2}}) < l(I_{x_{i+1}})$ and $r(I_{x_{i-1}}) \cap l(I_{x_{i+1}}) = \epsilon$ where i starts at i=4 and i=4t, t=1,2,......

Thus we have a unit probe interval representation of G.

■

Lemma 4.5.16 *A 2-snake is a unit probe interval graph.*

Proof:

Let us consider the 2-snake in Figure 4.39. The central vertices of all the 4-fans involved are probes. All the vertices of the edge-consecutive 4-fans that share exactly one 3-clique are also made probes. The unit probe interval representation of a 2-snake is given in Figure 4.39.

■

Lemma 4.5.17 *An extended staircase is a unit probe interval graph.*

Proof:

The unit probe interval representation of an extended staircase is given in Figure 4.43.

■

It must be noted here that merge of extended staircases with three 4-fans does not have any unit probe interval representation since they form an induced $F9$. It has a unit probe interval representation until the merge of two 4-fans but once the third 4-fan is merged we get an induced $F9$. Both an extended staircase-staircase-extended staircase-staircase and an extended staircase-staircase-extended staircase-staircase have unit probe interval representation, as shown in figure 4.40

Lemma 4.5.18 *Three edge-consecutive 4-fans that form a 2-path have unit probe interval representation if and only if they do not have F2, F7, F8, F9 as induced subgraphs.*

Proof: Let G_1, G_2, G_3 be three edge-consecutive 4-fans that form a 2-path. If they have a unit probe interval representation they cannot contain $F2, F7, F8, F9$ as induced subgraphs since these are forbidden subgraphs for unit probe interval graphs.

For the other direction let G_1, G_2, G_3 be three edge-consecutive 4-fans that form a 2-path with no $F2, F7, F8, F9$ as induced subgraphs. By lemma 4.5.6 they are one of the graphs in Figure 4.31. Parts 4, 5, 6 of Figure 4.31 form an induced $F9, F7$ and $F8$ respectively and hence they cannot be taken into consideration. So we have to provide the unit probe interval representation of parts 1, 2, 3 of the same Figure. Parts 1 and 2 are a staircase and a 2-snake whose unit probe interval representation are given in Figure 4.38 and Figure 4.39 respectively, and the unit probe interval representation of part 3 of the Figure 4.31 is given in Figure 4.43. ∎

Lemma 4.5.19 *Any number of edge-consecutive 4-fans that form a 2-path have unit probe interval representation if they do not have F2, F3, F4, F5, F6, F7, F8, F9, F10, F11 as generated subgraphs.*

Proof: Let G_1, G_2, G_3, ...G_n be n ($n > 2$) edge-consecutive 4-fans that form a 2-path with no $F2, F3, F4, F5, F6, F7, F8, F9, F10, F11$ as induced subgraphs.

$n = 3$: is true by the above lemma.

$n = 4$: Let G_1, G_2, G_3, G_4 be four edge consecutive 4-fans. We know that there are exactly three unit probe representable structures for three edge-consecutive 4-fans as shown in parts 1, 2, 3 of the Figure 4.31. Thus the merge of G_1, G_2, G_3 must be one of those three structures. G_2, G_3, G_4 also forms one of them. Thus corresponding to each of the three structures formed by G_1, G_2, G_3 we get three structures from G_2, G_3, G_4. Thus in total we have 10 structures, four of which are just continuations of 1, 2 and 3 of Figure 4.31. Since 3 is a non-symmetric structure its continuation can be done in 2 ways as shown in Figure 4.40 (one by repeating 3 and the other by doing 3 and reverse of 3 consecutively). We have a unit probe representation for the structures that are just continuations of the same type in the Figures 4.38, 4.39, 4.40 for any n. So we need to focus on structures which are combinations of different kinds namely 1, 2, 3 from Figure 4.31.

Figures 4.41, 4.42, 4.43 give the unit probe interval representation for three 4-fans G_1, G_2, G_3 where vertices $u_1, u_2, u_3, u_4, u_5, u_6$ are vertices forming the first 4-fan G1. If we add the fourth edge-consecutive 4-fan we get the structures in Figures 4.44, 4.45, 4.46 apart from the structures discussed above. We will consider three cases corresponding to Figures 4.44, 4.45, 4.46.

Case1: Figure 4.44 gives the possible structures obtained from Figure 4.41.

Case2: Figure 4.45 gives the possible structures obtained from Figure 4.42.

Case3: Figure 4.46 gives the possible structures obtained from Figure 4.43.

We look at Case 1 first. If we compare the representations in Figures 4.41 and 4.44 it is easy to see that the intervals of the vertices of G_1 remain unchanged in both the Figures implying that the addition of G_4 does not affect the intervals of vertices from G_1. So the removal and subsequent addition of G_1 from the representation will not affect the remaining representation of G_2, G_3, G_4. Furthermore it is also easy to see that the addition of G_1 implies adding at most 4 vertices which is the case when G_1 share an edge with G_2 and at least 2 vertices when G_1 shares exactly 2 K_3s with G_2. It can be seen from the way these different representations have been laid out that intervals of vertices from G_1-G_2 that get added do not overlap with the intervals of vertices from G_4-G_3. The same can be said about the other two cases.

$n = 5$: Let G_1, G_2, G_3, G_4, G_5 be five edge consecutive 4-fans. We will now get a representation of 5 five edge-consecutive 4-fans in an iterative way. We will first do the representation of G_3, G_4, G_5. Since we know that G_2 can be added to G_3, G_4 and G_5 without affecting their representation such that the bigger graph G_2, G_3, G_4, G_5 has a unit probe representation, we first add the intervals corresponding to the remaining vertices of G_2. Now we just consider G_2, G_3, G_4. Again, intervals from G_1 can be added to it without affecting the representation of G_2, G_3, G_4 such that G_1, G_2, G_3, G_4 has a unit probe interval representation. So we add the remaining intervals

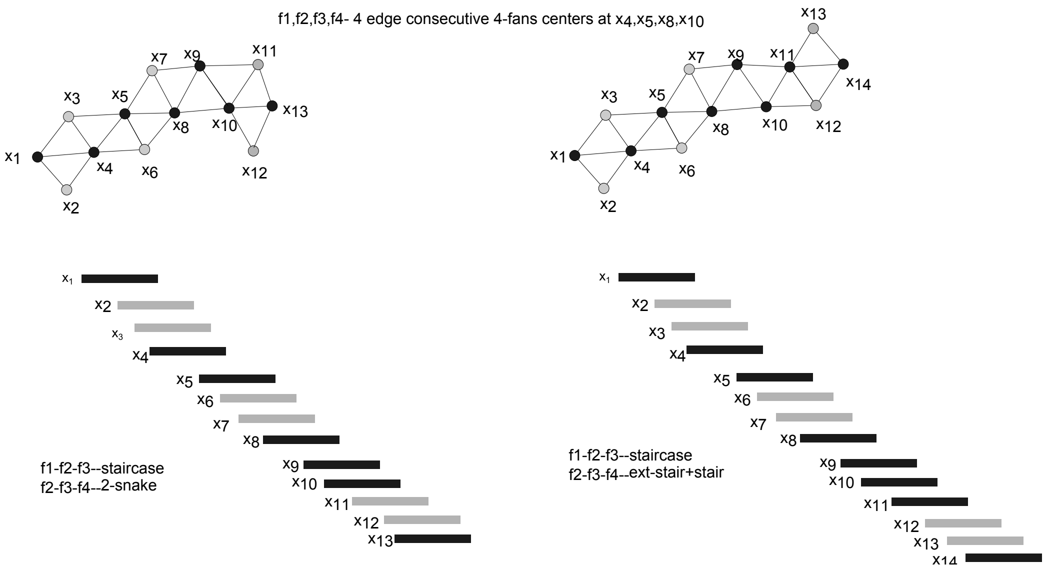


Figure 4.44: Four edge-consecutive 4-fans, structure 1

corresponding to the vertices of G_1 to get the unit probe interval representation of G_1, G_2, G_3, G_4, G_5. Also the intervals from G_1-G_2 does not overlap with the intervals from G_5-G_4. Hence again we can conclude that addition of G_1 does not affect the representation of G_2, G_3, G_4, G_5 such that G_1, G_2, G_3, G_4, G_5 has a unit probe interval representation.

In this way we can keep adding 4-fans to the present structures and obtain a unit probe interval representation at every step. Similarly we can show that this iterative process is valid for the the other two cases too. Hence we can conclude that any number of edge-consecutive 4-fans have unit probe interval representation if they do not have $F2, F7, F8, F9$ as induced subgraphs.

■

Based on the previous discussions we can think of a 2-path as a straight 2-path or bundles of 4-fans (edge or vertex consecutive) or their merge. So we will look at a 2-path as merge of vertex or edge consecutive 4-fans joined together by straight 2-paths. Thus a 2-path can be of three types:

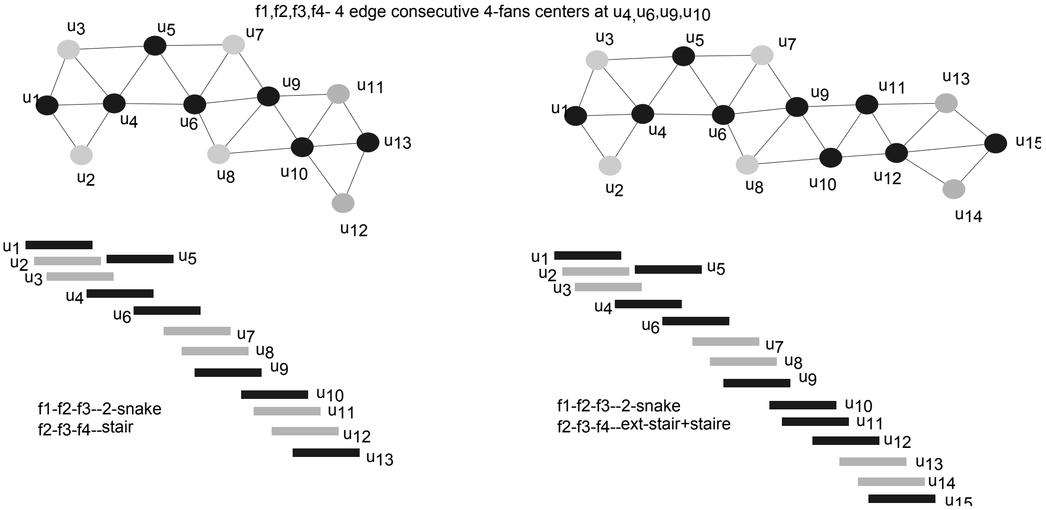

f1,f2,f3,f4- 4 edge consecutive 4-fans centers at u4,u6,u9,u10
f1-f2-f3--2-snake
f2-f3-f4--stair
f1-f2-f3--2-snake
f2-f3-f4--ext-stair+staire

Figure 4.45: Four edge-consecutive 4-fans, structure 2

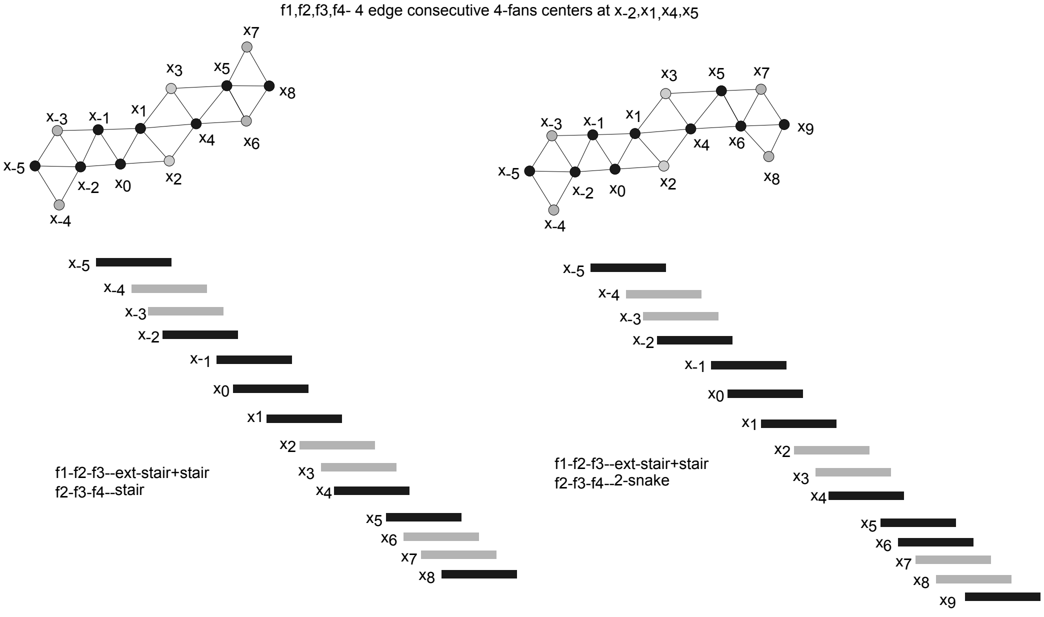

f1,f2,f3,f4- 4 edge consecutive 4-fans centers at x-2,x1,x4,x5
f1-f2-f3--ext-stair+stair
f2-f3-f4--stair
f1-f2-f3--ext-stair+stair
f2-f3-f4--2-snake

Figure 4.46: Four edge-consecutive 4-fans, structure 3

1.Groups of edge-consecutive 4-fans joined by straight 2-paths.

2.Groups of vertex-consecutive 4-fans joined by straight 2-paths.

3.A Group of edge-consecutive 4-fans joined by straight 2-path with a group of edge-consecutive 4-fans.

Thus we first show that each of the graphs obtained from 1, 2, and 3 can have a unit probe interval representation if they do not have $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as generated subgraphs.

Theorem 4.5.20 *Groups of edge-consecutive 4-fans joined by straight 2-paths have unit probe interval representation if they do not have F2, F3, F4, F5, F6, F7, F8, F9, F10 or F11 as generated subgraphs.*

Proof: Let $B_1, B_2, B_3, ...B_m$ be m $(m \geq 2)$ groups of edge-consecutive 4-fans that form a 2-path with no $F2, F3, F4, F5, F6, F7, F8, F9, F10, F11$ as induced subgraphs and they are joined by straight 2-paths of length n. Let us call this structure G. We will now give a unit probe interval representation of G.

Let B_i and B_j be any two groups of 4-fans joined by a straight 2-path of length n. Let us call this B_{ij}. Since the representation of vertices located on one end of B_i is independent of the representation of vertices on the other end of B_i for any B_i (as seen from Figure 4.27 and 4.34). So we can conclude that if the theorem holds for B_{ij} it should hold for the entire G. So we give a unit probe interval representation of B_{ij} for different values of n.

Case 1

$n = 0$: This is the case where the intermittent straight 2-path is of length 0. This implies that we have a merge of B_i and B_j such that the last edge of B_i coincides with the first edge of B_j. Both B_i and B_j are groups of edge consecutive 4-fans. So their merge in this fashion gives a lengthier sequence of edge-consecutive 4-fans whose length is equal to the sum of the lengths of B_i and B_j. By the previous lemma B_{ij}

must have a unit probe interval representation.

Case 2

$n = 1$: This is the case where the intermittent straight 2-path is of length 1. Both B_i and B_j are groups of edge consecutive 4-fans. So we get 2 sequences of edge-consecutive 4-fans joined by a straight 2-path of length one as shown in case $n = 1$ in Figure 4.47. From representations of the four edge-consecutive 4-fans as shown in Figures 4.44, 4.45, 4.46 it is easy to see that the last added vertices of the 2-paths are non-probes and their intervals can be made to have distinct right end points. Here the last vertex of B_i is either the first vertex of B_j or they are adjacent as shown in Figure 4.48.

Let us first consider the case when the last vertex of B_i is the first vertex of B_j. If v_{n-1} and v_n are the last two added vertices of B_i then in the representation $r(I_{v_{n-1}}) < r(I_{v_n})$ or $r(I_{v_n}) < r(I_{v_{n-1}})$ and v_n is a non-probe, v_{n-1} is a probe. If w_1 and w_2 are the vertices of the first K_2 in B_j then in the representation $r(I_{w_1}) < r(I_{w_2})$ or $r(I_{w_1}) < r(I_{w_2})$ and w_1 is a non-probe and w_2 is a probe. Thus we can do unit probe representation of B_{ij} by doing B_i first and then continuing with the intervals B_j by making the last interval of B_i become the first interval of B_j such that $r(I_{v_{n-1}}) < r(I_{v_n=w_1}) < r(I_{w_2})$ and $I_{v_{n-1}} \cap I_{w_2} \neq \phi$.

Now let us consider that the last vertex of B_i is adjacent to the first vertex of B_j. In this case we either get induced $F2$ which is avoided or merge involving edge consecutive 4-fans which also do not have unit probe interval representation if the two end 2-leaves of B_i and B_j are forced to be non-probes. For such a case we get induced forbidden subgraphs $F2, F7, F8$ as illustrated in the Figures 4.49, 4.50. The only way a unit probe interval representation is possible in this subcase is illustrated in the last row of the same Figure 4.50 with its representation.

Case 3

$n = 2$: This is the case where the intermittent straight 2-path is of length 2.

Both B_i and B_j are groups of edge consecutive 4-fans. Length of B_{ij} is equal to 2+length(B_i)+length(B_j)as shown in case $n = 2$ in Figure 4.47. We are interested in the last two 4-fans of B_i and the first two 4-fans of B_j. From the representations in Figure 4.27 it is easy to see that the end 2-leaves of extended-staircase and 2-snake have non-probe restrictions as shown in second and third picture of Figure 4.27. Thus the only structure which has no non-probe restriction on the 2-leaf on one end is the staircase as shown in Figure 4.27. Furthermore the 2-leaf on that end can be made to have a distinct end point. Thus only way there could be problems in the representation if B_i=2-snake, extended staircase and B_j=2-snake, extended staircase. All possible structures obtained in this case are given in Figures 4.51, 4.52, each one of which is either a forbidden subgraph or has a forbidden subgraph as an induced subgraph. The only possible structure that can be formed in this situation is if right end 2-leaf from B_i is non-adjacent to the left end 2-leaf of B_j as shown in the Figure 4.53. So we look at the case when one of B_i or B_j is a staircase. Without loss of generality let B_i end in a staircase. The different ways in which B_j can be added to it via a straight 2-path of length 2 are given in Figure 4.54. Let v_n be the 2-leaf in the right end of B_i and v_{n-1} is the vertex added just before v_n. V_n can be either probe or non-probe. If v_n is a probe then by the second representation of the staircase, I_{v_n} has a distinct right end point and $r(I_{v_{n-1}}) < r(I_{v_n})$. If v_n is a non-probe then by the first representation of the staircase, either $I_{v_{n-1}}$ or I_{v_n} has a distinct right end point and $r(I_{v_n}) < r(I_{v_{n-1}})$ or $r(I_{v_{n-1}}) < r(I_{v_n})$. The possible representations are given in Figure 4.54. Since v_n can be either probe or non-probe and furthermore the 2-leaves at the left end of all the three namely, staircase, 2-snake and extended staircase can be made to have a distinct left end points such that if u_1 is the left end 2-leaf, u_2 be the other vertex of the first K_2 and u_3 be the third added vertex such that u_3 is the center of the first K_3 of B_j then either $r(I_{u_1}) < r(I_{u_2}) < r(I_{u_3})$ or $r(I_{u_2}) < r(I_{u_1}) < r(I_{u_3})$ (by the Figure 4.27), so easily the second, third and last addition works fine for the

Figure 4.54. The only non-possibility here is the subcase in picture 1 of the same figure. This is the case when v_n is adjacent to u_1 where u_1 is the left end 2-leaf of B_j and u_1 is a part of a 2-snake or extended staircase. In other words u_1 is forced to be a non-probe so that v_n is forced to be a probe and so $r(I_{v_{n-1}}) < r(I_{v_n})$ but $u_2 \leftrightarrow v_{n-1}$ and $u_2 \not\leftrightarrow v_n$. In this case several forbidden subgraphs are formed which are given in Figure 4.55 and so we do not have to consider them.

Case 4

$n = 3$: This is the case where the intermittent straight 2-path is of length 3. Length of B_{ij} is equal to 3+length(B_i)+length(B_j) as shown in case $n = 3$ in the Figure 4.47. Let v_n be the 2-leaf in the right end of B_i and v_{n-1} is the vertex added just before v_n. Let u_1 be the 2-leaf in the left end of B_j and u_2 be the vertex added just after u_1 as shown in the Figure 4.56. The only non-possibility here could be the case when both v_n and u_1 are adjacent and non-probes. In this case only one representation is possible which we get when B_i is a staircase since v_n can be made a probe here and $r(I_{v_{n-1}}) < r(I_{v_n})$. It is illustrated in Figure 4.57.

So the problem arises when B_i, B_j are 2-snake or extended-staircase. All the possible structures in this case are given in Figure 4.58 all of which are forbidden subgraphs.

Case 5

n >3: This is the case where the intermittent straight 2-path is of length more than 3. Length of B_{ij} is greater than 3+length(B_i)+length(B_j) as shown in case $n = 4$ in Figure 4.47. In this case the two end 2-leaves, in fact the two end K_2s are not adjacent since there is at least one vertex between them. Moreover all the vertices of the straight 2-path also have no probe-non-probe restriction on them. Thus we can easily do a unit probe representation B_{ij} by doing B_i first and then continuing with the intervals of the straight 2-path followed by the intervals of vertices from B_j.

■

Lemma 4.5.21 *Three vertex-consecutive 4-fans that form a 2-path have a unit probe*

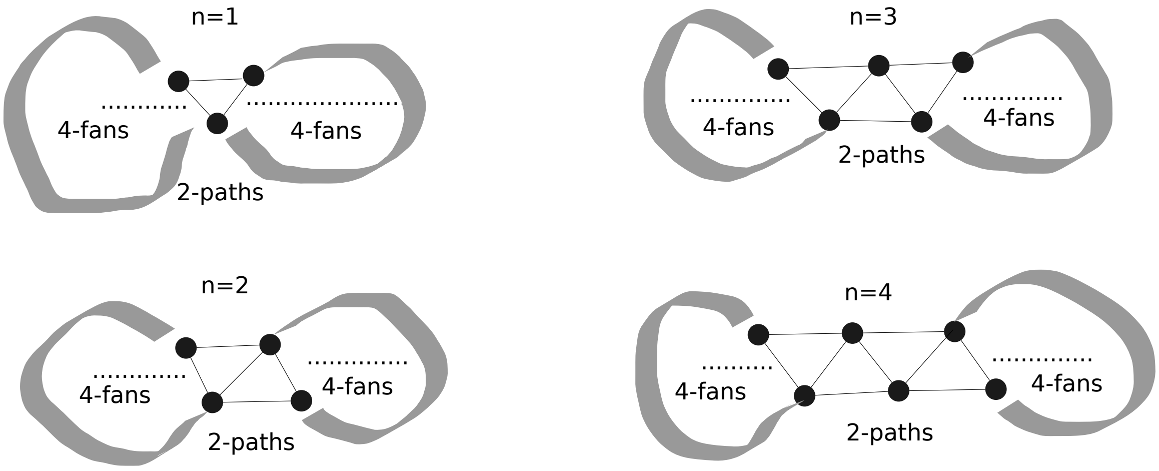


Figure 4.47: Example of merges for different values of n

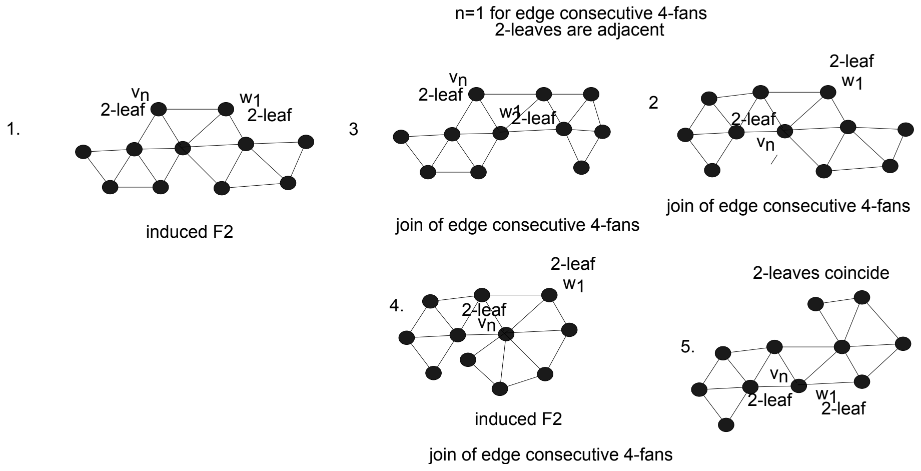


Figure 4.48: $n = 1$: edge-consecutive merge for 2-paths

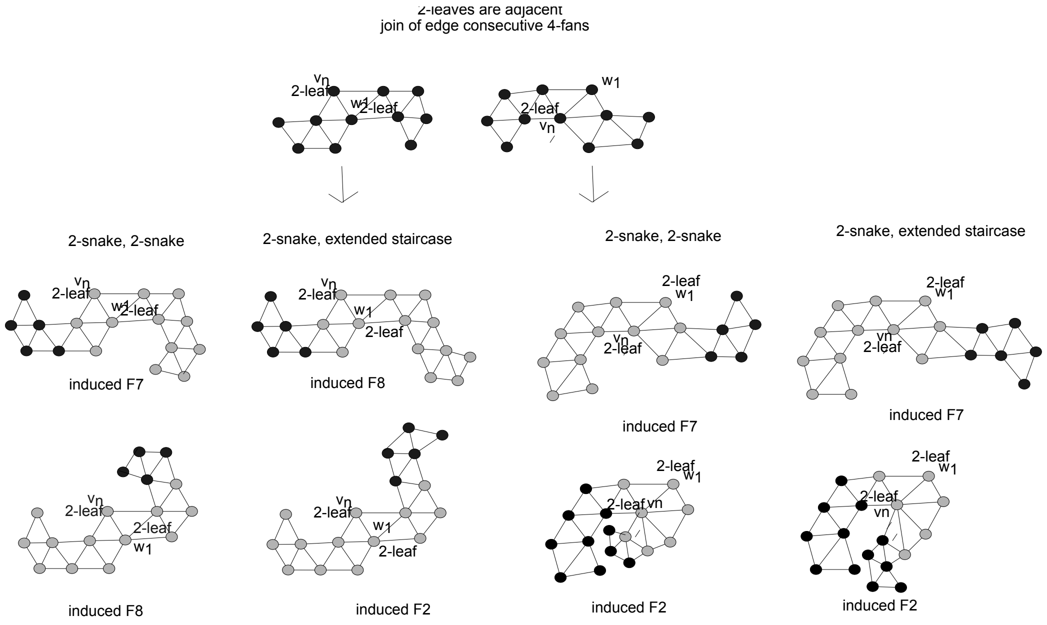


Figure 4.49: $n = 1$: edge-consecutive merge for 2-paths

interval representation if and only if they do not have F2 as a generated subgraph.

Proof: Let G_1, G_2, G_3 be three vertex-consecutive 4-fans. Let it be a unit probe interval graph. Therefore it cannot conatain $F2$ as generated subgraph. Let G_1, G_2, G_3 be three vertex-consecutive 4-fans that form a 2-path with no $F2$ as induced subgraphs. So it must be the graph in Figure 4.34. The same picture gives its unit probe interval representation. ■

Note: we just consider $F2$ for the above lemma because it is the only structure which cropped up during the construction of vertex-consecutive 4-fans as seen earlier.

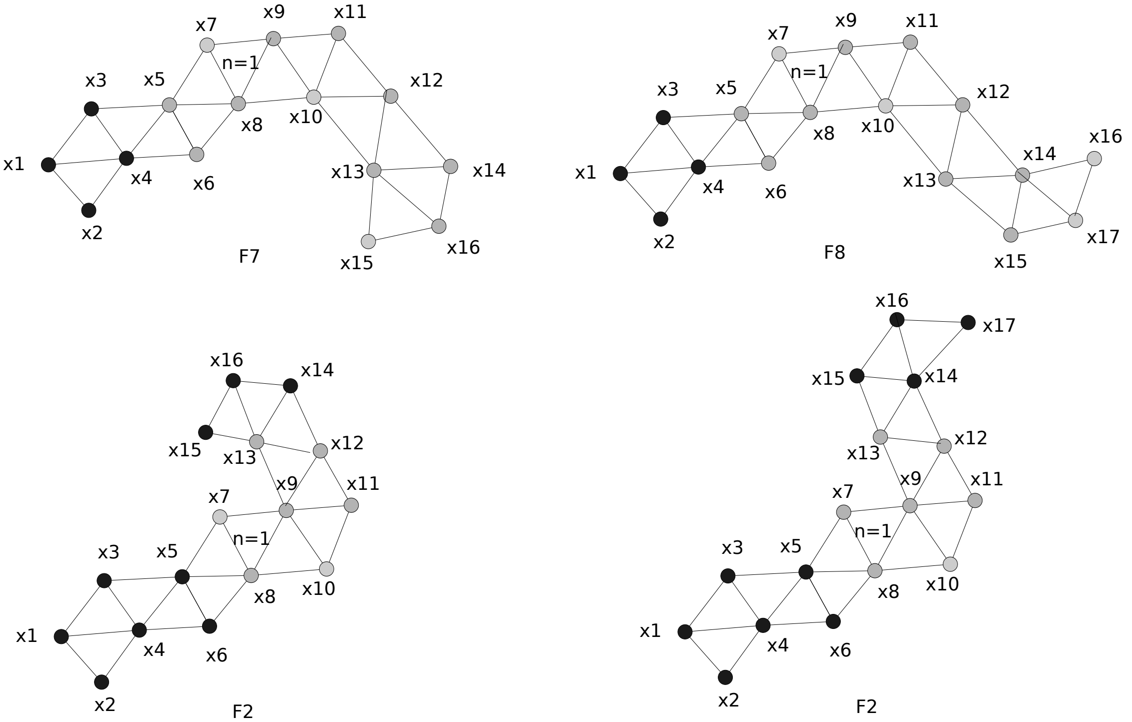
Lemma 4.5.22 *Any number of vertex-consecutive 4-fans that form a 2-path without F2 as a generated subgraph have a unit probe interval representation.*

Proof: Let $G_1, G_2, G_3, ...G_n$ be n $(n > 2)$ vertex-consecutive 4-fans that form a 2-path with no $F2$ as induced subgraphs. We know that every set of 3 vertex consecutive 4-fans, say, A_1=G_1, G_2, G_3, A_2=G_4, G_5, G_6,..., A_k=G_{i-1}, G_i, G_{i+1},.... has a unit probe representation similar to that in Figure 4.34. Since all the A_j's $j =$

x8 is the 2-leaf of Bj (if x9 is the 2-leaf then we always get an F2)

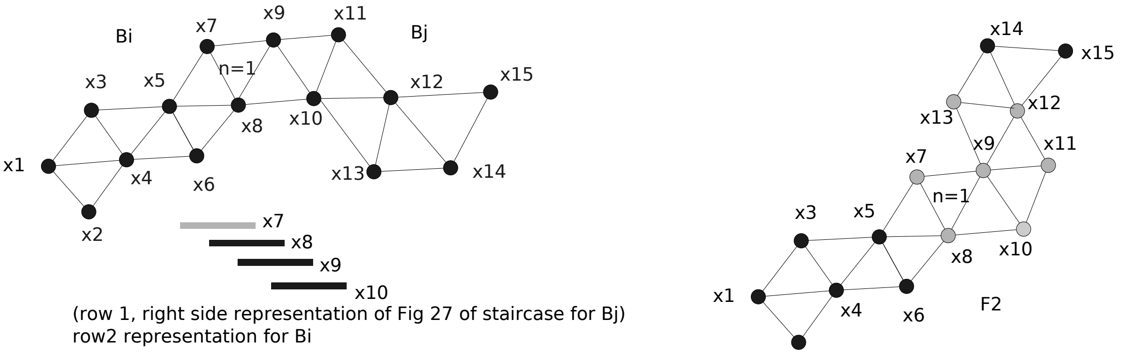
staircase+2-snake

staircase+extended staircase



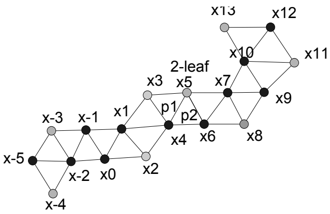
staircase+ staircase

staircase+ staircase



(row 1, right side representation of Fig 27 of staircase for Bj)
row2 representation for Bi

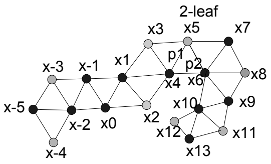
Figure 4.50: $n = 1$: edge-consecutive merge for 2-paths



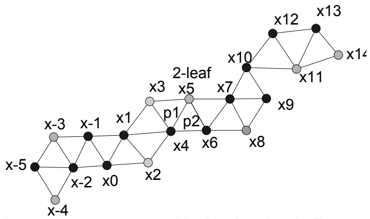
x2-x1-x4-x3-x5-x6-x7-x8-x9-x10-x11-x12-x13 Forms an F8

ext-stair+ 2-snake

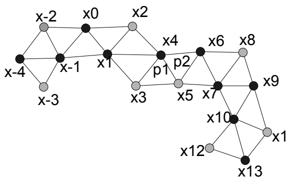
x5- is the 2-leaf (the edge between the 2-leaves is an non-inner edge)-first way of p1-p2



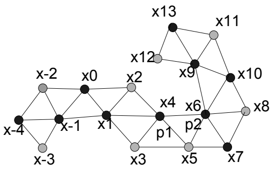
x4-x5-x6-x7-x8-x9-x10 Forms an F2
ext-stair+ 2-snake



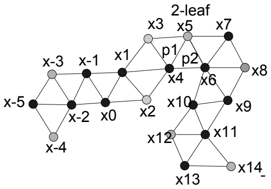
x2-x1-x4-x3-x5-x6-x7-x8-x9-x10-x11-x12-x13-x14 Forms
ext-stair+ ext-stair



x2-x1-x4-x3-x5-x6-x7-x8-x9-x10-x11-x12-x13 Forms an F8
2-snake + 2-snake

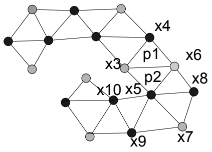


x2-x1-x4-x3-x5-x6-x7-x8-x9-x10-x11-x12-x13 Forms an F7
2-snake + 2-snake

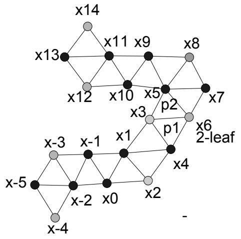


x4-x5-x6-x7-x8-x9-x10 Forms an F2
ext-stair+ ext-stair

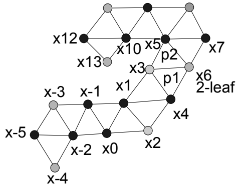
x6 is the 2-leaf (edge bertween the 2-leaves is an inner edge)-second way of p1-p2



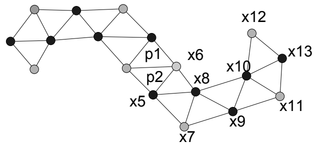
x3-x5-x6-x7-x8-x9-x10 forms an F2
2-snake+2-snake



x3-x5-x6-x7-x8-x9-x10 forms an F2
ext-stair+ ext-stair

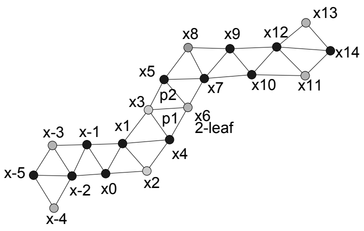


x3-x5-x6-x7-x8-x9-x10 forms an F2
ext-stair+ 2-snake

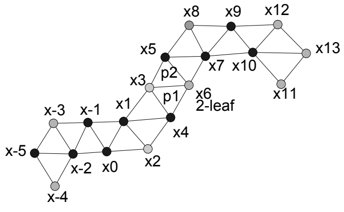


F3

2-snake+2-snake



F10
ext-stair+ ext-stair



F11
ext-stair+ 2-snake

Figure 4.51: $n = 2$: edge-consecutive merge for 2-paths

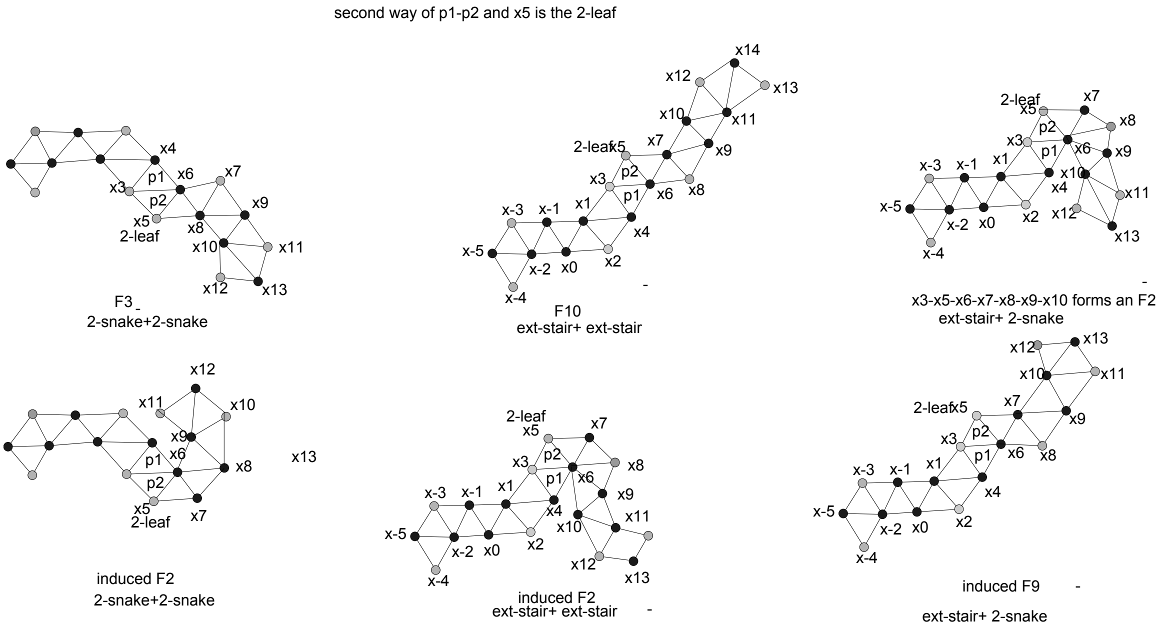


Figure 4.52: $n = 2:$ edge-consecutive merge for 2-paths

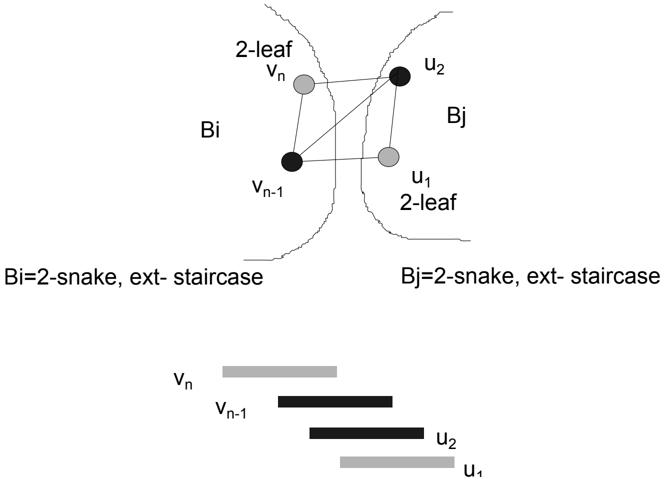


Figure 4.53: $n = 2:$ edge-consecutive merge for 2-paths

vn-last added vertex of a staircase -> vn is either a probe or non-probe

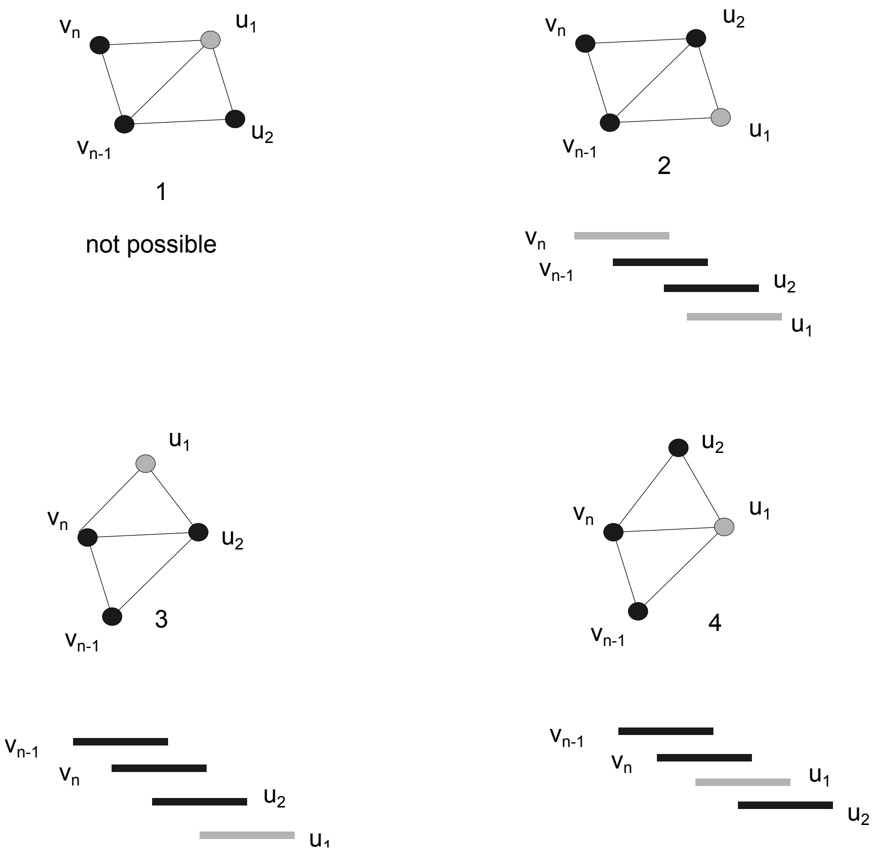
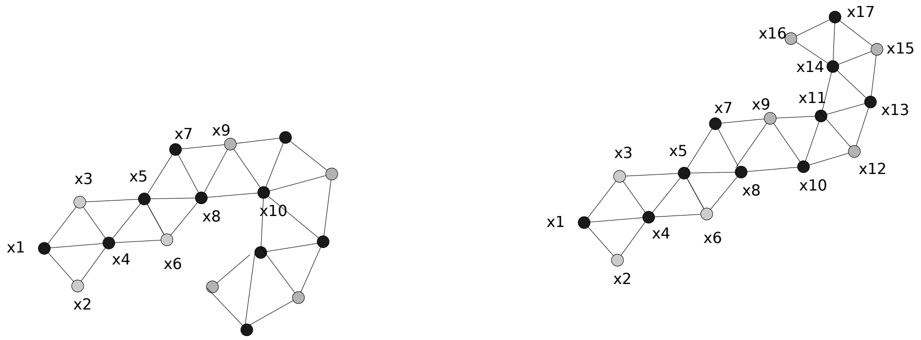
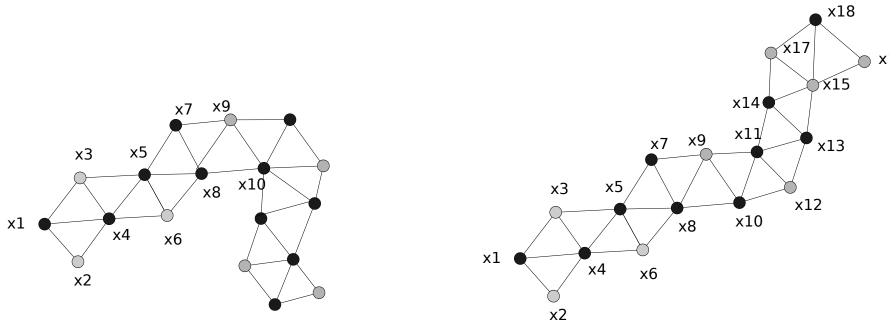


Figure 4.54: $n = 2$: edge-consecutive merge for 2-paths



induced F5 with center at x10

x5-x6-x8-.........x14-x15-x17-x16 forms an induced F8



x5-x6-x8-.........x14-x15-x17-x16 forms an induced F9

induced F5 with center at x10

Figure 4.55: $n = 2$: edge-consecutive merge for 2-paths

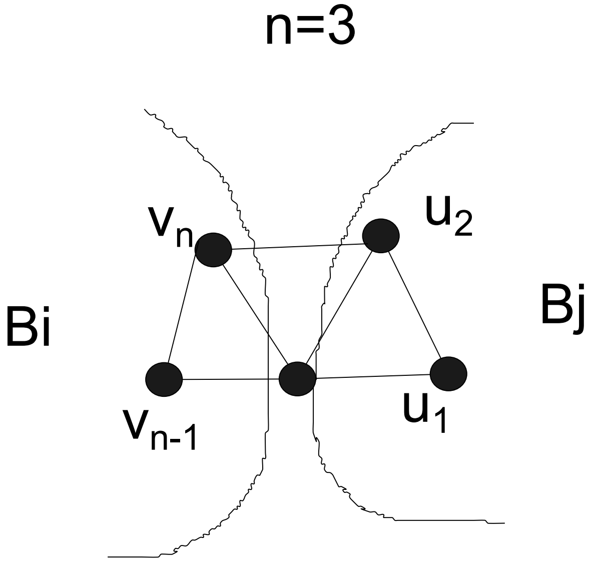


Figure 4.56: $n = 3$: edge-consecutive merge for 2-paths

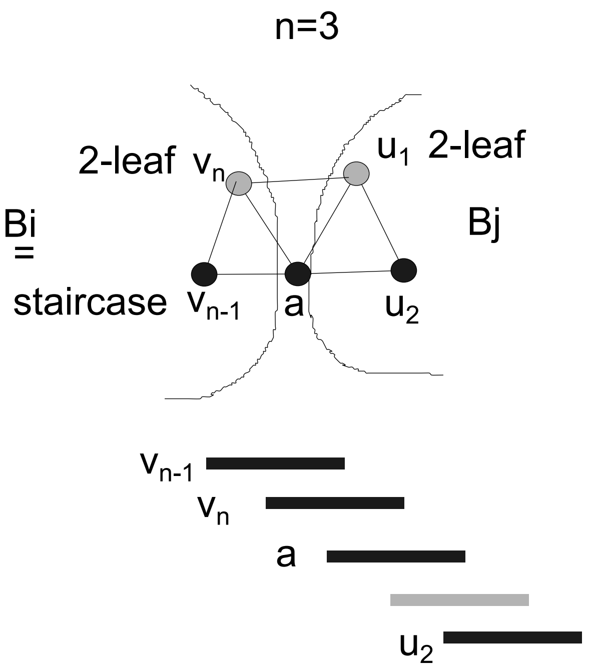
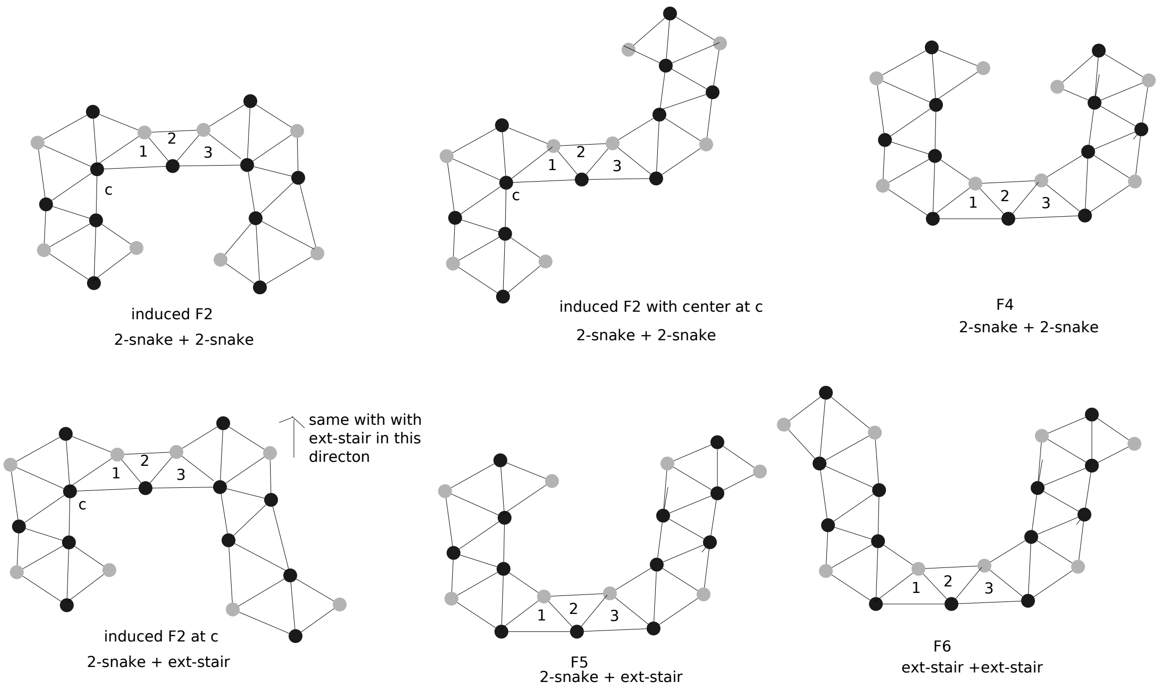


Figure 4.57: $n = 3$: edge-consecutive merge for 2-paths



induced F2
2-snake + 2-snake

induced F2 with center at c
2-snake + 2-snake

F4
2-snake + 2-snake

induced F2 at c
2-snake + ext-stair

F5
2-snake + ext-stair

F6
ext-stair +ext-stair

Figure 4.58: $n = 3$: edge-consecutive merge for 2-paths

$1, 2, ...k, ...$ are again vertex consecutive, they all share the end vertices. Thus the right end 2-leaf of A_j is the left end 2-leaf of A_{j+1}. Hence the last 2 added vertices of A_j must be adjacent to the first K_2 of A_{j+1} such that the last and first vertex coincides. Let v_i, v_{i-1}, v_{i-2} be the last 3 added vertices of A_j. Thus if v_i is the right end 2-leaf A_j then v_i must also be the left end 2-leaf of A_{j+1}. Hence v_i and v_{i+1} are the vertices of the first K_2 of A_{j+1}. Let v_{i+2} be third added vertex of A_{j+1}. It is easy to see that v_{i-1}, v_i, v_{i+1} forms a K_3. According to a representation 1 in Figure 4.34, v_{i-1} and v_{i+1} are probes and v_i is non-probe. Moreover $r(I_{v_{i-2}}) < r(I_{v_{i-1}}) < r(I_{v_i})$ and $r(I_{v_i}) < r(I_{v_{i+1}}) < r(I_{v_{i+2}})$. Thus we easily paste the representations of the sets of A_i's one after another such that $r(I_{v_{i-1}}) < r(I_{v_i}) < r(I_{v_{i+1}})$ to get a unit probe representation. This same scheme can be extended for all the A_i's to get a unit probe interval representation for the entire graph. ■

Theorem 4.5.23 *Groups of vertex-consecutive 4-fans joined by straight 2-paths have a unit probe interval representation if they do not have F2 as a generated subgraph.*

Proof: Let $B_1, B_2, B_3, ...B_m$ be m ($m \geq 2$) groups of vertex-consecutive 4-fans that form a 2-path with no $F2$ as induced subgraph and suppose they are joined by straight 2-paths of length n. Let us call this structure G. We will now give a unit probe interval representation of G.

Let B_i and B_j be any two groups of 4-fans joined by a straight 2-path of length n. Let us call this B_{ij}. Since the representation of vertices located on one end of B_i is independent of the representation of vertices on the other end of B_i for any B_i (as seen from Figure 4.34), we can conclude that if the theorem holds for B_{ij} it should hold for the entire G. So we give a unit probe interval representation of B_{ij} for different values of n.

Case 1

$n = 0$: This is the case where the intermittent straight 2-path is of length 0. This

implies that we have a merge of B_i and B_j such that the last edge of B_i coincides with the first edge of B_j. Both B_i and B_j are groups of vertex-consecutive 4-fans. Hence the last 2 added vertices of B_i must be the first K_2 of B_j. Let v_{n-1} and v_n be the last 2 added vertices of B_i and u_1, u_2 be the first K_2 of B_j such that v_n is the right end 2-leaf of B_i and u_1 is the left end 2-leaf of B_j. Given B_i there are 4 ways in which B_j can be added to it in this case as shown in the Figure 4.59. The first two are the subcases when the right end 2-leaf of B_i is adjacent to the left end 2-leaf of B_j and the second two are the subcases when both the 2-leaves coincide. Pictures 2 and 4 are not taken into consideration since they form an induced F2. For picture 1 we use the representation used in the figure 4.35. This is ok since right end 2-leaf of B_i end with distinct right end point, left end 2-leaf of B_j have a distinct left end point and both of them are probes. For picture 3 we make both the right end 2-leaf of B_i and the left end 2-leaf of B_j as non-probes. This can be done easily since both of them are same here. Then we use representation 1 of the Figure 4.34 for B_i and representation 2 of the Figure 4.34 for B_j.

Case 2

$n = 1$: This is the case where the intermittent straight 2-path is of length 1. This implies that we have B_i and B_j such that the last vertex or right end 2-leaf of B_i coincides with the left end 2-leaf of B_j or they are adjacent. Both B_i and B_j are groups of vertex-consecutive 4-fans. So in the first case we either get a lengthier sequence of vertex-consecutive 4-fans whose length is equal to the sum of the lengths of B_i and B_j plus one or an $F2$. Since $F2$ is avoided and by the previous lemma we know that B_{ij} must have a unit probe interval representation. In the second case too we either get an F2 which is avoided or a structure that has a unit probe interval representation as shown in the Figure 4.60

Case 3

$n = 2$: This is the case where the intermittent straight 2-path is of length 2. This

implies that we have B_i and B_j such that B_i and B_j shares nothing and the right end 2-leaf of B_i is either adjacent to the left end 2-leaf of B_j or it is adjacent to the other vertex of the first K_2 of B_j. Both B_i and B_j are groups of vertex-consecutive 4-fans. So we get 2 sequences of vertex-consecutive 4-fans joined by a straight 2-path of length 2. Length of B_{ij} is equal to 2+length(B_i)+length(B_j). Let v_n be the right end 2-leaf of B_i and u_1 be the left end 2-leaf of B_j such that v_{n-2}, v_{n-1} and v_n be the last 3 added vertices of B_i and u_1, u_2, u_3 be the first 3 added vertices of B_j where u_1, u_2 is the first K_2 of B_j. Given B_i, the total possible ways in which B_j can be added to B_i is illustrated in the Figure 4.61 and all of them have unit probe interval representations as shown in the same Figure.

Case 4

$n = 3$: This is the case where the intermittent straight 2-path is of length 3. This implies that we have B_i and B_j such that B_i and B_j shares nothing and we have 2 sequences of vertex-consecutive 4-fans joined by a straight 2-path of length 3. Let v_n be the 2-leaf in the right end of B_i and v_{n-1} is the vertex added just before v_n. let u_1 be the 2-leaf in the left end of B_j and u_2 is the vertex added just after u_1. The only non-possibility here could be the case when both v_n and u_1, the both 2-leaves are adjacent which turns out to be ok since v_n can be made a probe such that interval of v_n has a right distinct end point and u_1 a non-probe such that interval of u_1 has a left distinct endpoint as illustrated in the Figure 4.62.

Case 5

$n > 3$: This is the case where the intermittent straight 2-path is of length more than 3. This implies that we have B_i and B_j such that B_i and B_j shares nothing and we get 2 sequences of vertex-consecutive 4-fans joined by a straight 2-path of length at least 4. Length of B_{ij} is greater than 3+length(B_i)+length(B_j). In this case the two end 2-leaves, in fact the two end K_2s are not adjacent since there is atleast one vertex between them. Moreover all the vertices of the straight 2-path also have no

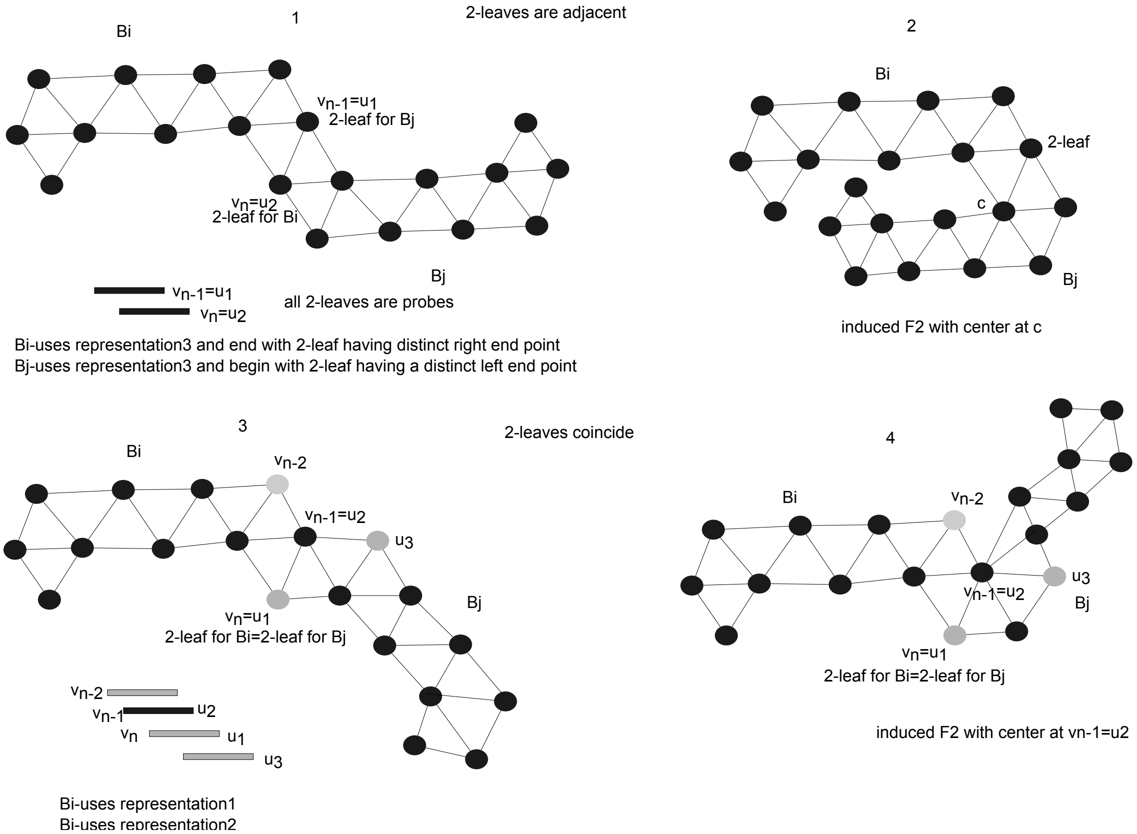

1
2-leaves are adjacent
Bi
vn-1=u1
2-leaf for Bj
vn=u2
2-leaf for Bi
Bj
vn-1=u1
vn=u2
all 2-leaves are probes
Bi-uses representation3 and end with 2-leaf having distinct right end point
Bj-uses representation3 and begin with 2-leaf having a distinct left end point
2
Bi
2-leaf
c
Bj
induced F2 with center at c
3
2-leaves coincide
Bi
vn-2
vn-1=u2
u3
Bj
vn=u1
2-leaf for Bi=2-leaf for Bj
vn-2
vn-1
u2
vn
u1
u3
Bi-uses representation1
Bi-uses representation2
4
Bi
vn-2
u3
vn-1=u2
Bj
vn=u1
2-leaf for Bi=2-leaf for Bj
induced F2 with center at vn-1=u2

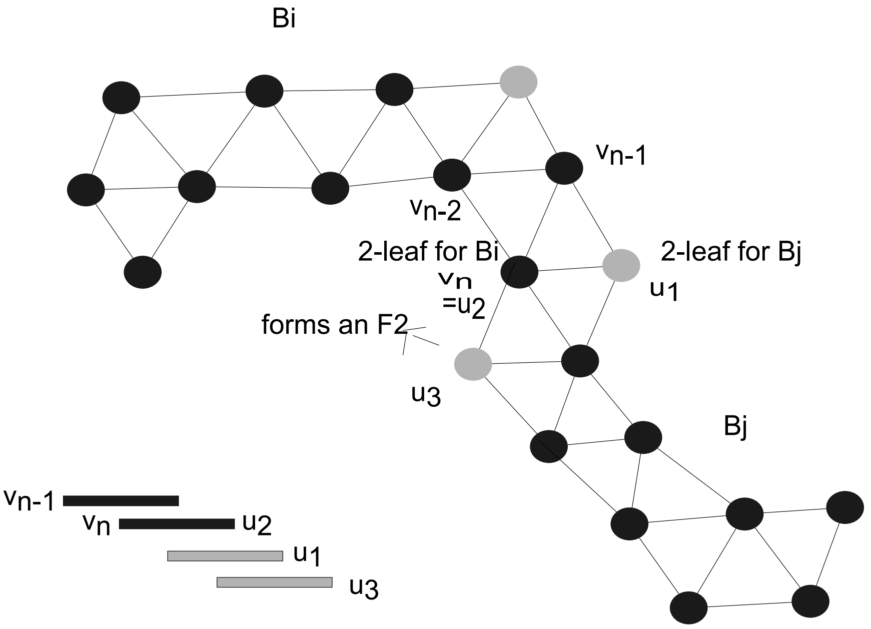
Figure 4.59: $n = 0$: vertex-consecutive merge for 2-paths

probe-non-probe restriction on it. Thus we can easily do unit probe representation of B_{ij} by doing B_i first and then continuing with the intervals of the straight 2-path followed by the intervals of vertices from B_j.

■

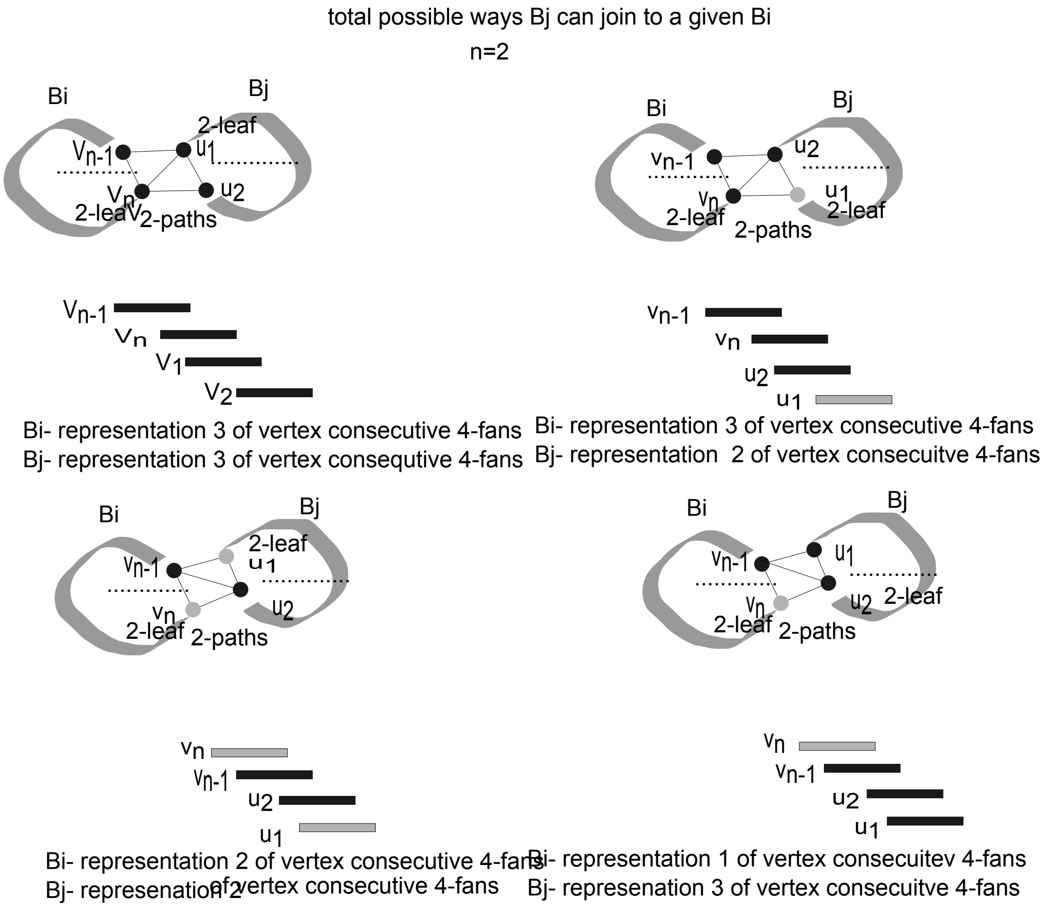
Theorem 4.5.24 *A group of vertex-consecutive 4-fans joined to a group of edge-consecutive 4-fans via a straight 2-path of length n (n ≥ 0) have a unit probe interval representation if they do not have F2, F3, F4, F5, F6 F7,F8, F9, F10, F11 as generated subgraphs.*

Proof:



Bi-uses representation of Figure 33 and end with 2-leaf having distinct right end point
Bj-uses representation2 of Figure 32

Figure 4.60: $n = 1$: vertex-consecutive merge for 2-paths



Bi- representation 3 of vertex consecutive 4-fans
Bj- representation 3 of vertex consequtive 4-fans

Bi- representation 3 of vertex consecutive 4-fans
Bj- representation 2 of vertex consecuitve 4-fans

Bi- representation 2 of vertex consecutive 4-fans
Bj- represenation 2 of vertex consecutive 4-fans

Bi- representation 1 of vertex consecuitev 4-fans
Bj- represenation 3 of vertex consecuitve 4-fans

Figure 4.61: $n = 2$: vertex-consecutive merge for 2-paths

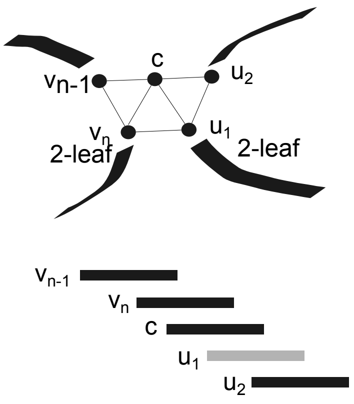


Figure 4.62: $n = 3$: vertex-consecutive merge for 2-paths

Let G_1 and G_2 be groups of vertex-consecutive and edge-consecutive 4-fans respectively. We know that any number of edge-consecutive 4-fans that form a 2-path have unit probe interval representation if they do not have $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. We also know that any number of groups of vertex-consecutive 4-fans have unit probe interval representation if they do not have $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. Let v_{n-1} and v_n be the last 2 added vertices of G_1 and u_1, u_2 be the first K_2 of G_2 such that v_n is the right end 2-leaf of G_1 and u_1 is the left end 2-leaf of G_2.

Case 1 ($n = 0$):

The last 2 added vertices of G_1 must be the first K_2 of G_2.

If v_n is adjacent to u_1 then we get problem in the representation for G_2= 2-snake or extended stair, since in these case $v_{n-1} = u_1$ and u_1 is forced to be a non-probe from G_2, but v_{n-1} is a probe from G_1. We get 4 structures here all of which have an induced forbidden subgraph in it as shown in Figure 4.63. This case is ok with G_2=staircase (and we avoid the formation of 5-fans) since u_1 can be made a probe too in a staircase and we use the representation 3 for vertex consecutive 4-fans to represent G_1 such that v_{n-1} and v_n are probe and v_n has a distinct right end point as shown in Figure 4.64.

If v_n coincides with u_1 then we are good as shown in picture 5 of the Figure 4.63 and in the Figure 4.64.

Case 2 ($n = 1$):

There exists exactly one K_3 between G_1 and G_2. We have two cases here. Either the 2-leaves are adjacent or they coincide which are both ok as shown in figure 4.65.

Case 3 ($n = 2$):

There exists two K_3's between G_1 and G_2. The possible structures and their representations are given in Figure 4.66. The only non-possibility is when G_2= 2-snake, extended staircase where the the 2-leaves are always non-probes and we get forbidden

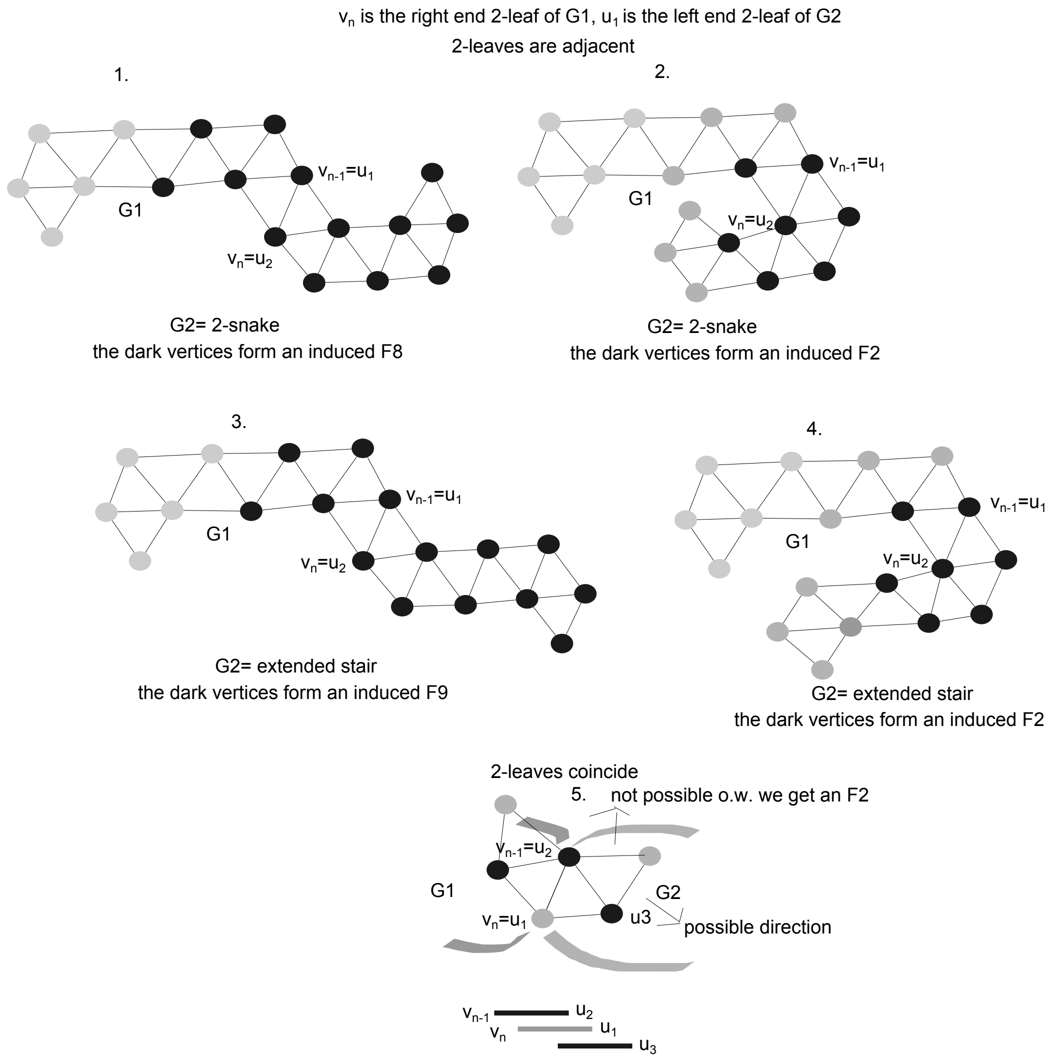

v_n is the right end 2-leaf of G1, u_1 is the left end 2-leaf of G2
2-leaves are adjacent
1.
v_n-1=u_1
G1
v_n=u_2
G2= 2-snake
the dark vertices form an induced F8
2.
v_n-1=u_1
G1
v_n=u_2
G2= 2-snake
the dark vertices form an induced F2
3.
v_n-1=u_1
G1
v_n=u_2
G2= extended stair
the dark vertices form an induced F9
4.
v_n-1=u_1
G1
v_n=u_2
G2= extended stair
the dark vertices form an induced F2
2-leaves coincide
5.
not possible o.w. we get an F2
v_n-1=u_2
G1
v_n=u_1
u3
G2
possible direction
v_n-1
u_2
v_n
u_1
u_3

Figure 4.63: vertex-consecutive-edge-consecutive for 2-path

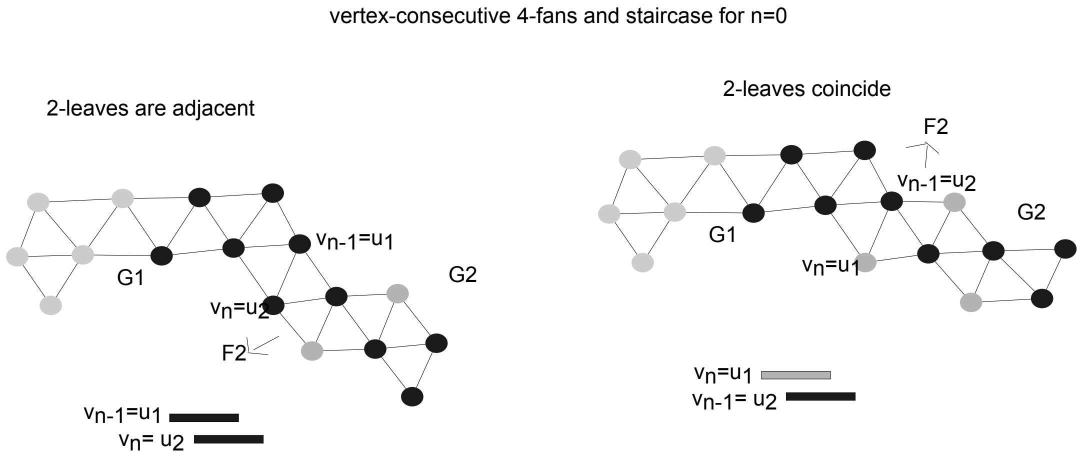
subgraphs $F2, F8, F9$ as shown in Figure 4.67.

Case 4 ($n \geq 3$):

For $n \geq 3$ we use the same argument as we have used for the vertex consecutive case and it works fine.

■

The following Lemma will prove that certain 2-paths are always unit probe interval graphs. After this we will generalize it further and continue to prove that any 2-path will have a unit probe interval representation if and only if it does not contain



here v_{n-1} is made a probe and by the 3 rd representation of staircase
it must have distinct left end point
and vn is also made a probe and by the 3 rd representation of vertex consecutive 4-fans
it must have distinct right end point

2-leaf v_n is made a non-probe and
by the 2 nd representation of vertex consecutive 4-fans
it can have non-distinct right end point and
by the staircase representation it it can have distinct
left end point

Figure 4.64: vertex-consecutive-edge-consecutive for 2-path

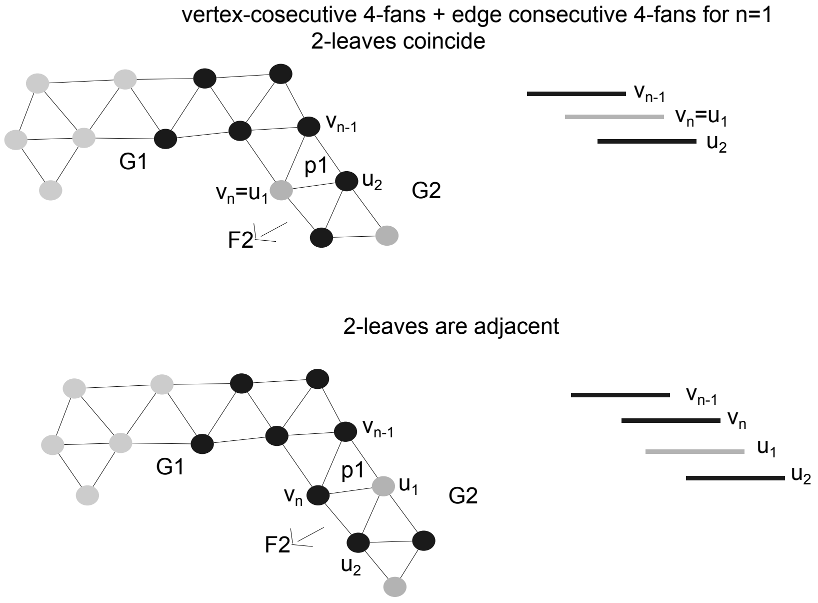


Figure 4.65: $n = 1$: vertex-consecutive-edge-consecutive for 2-path

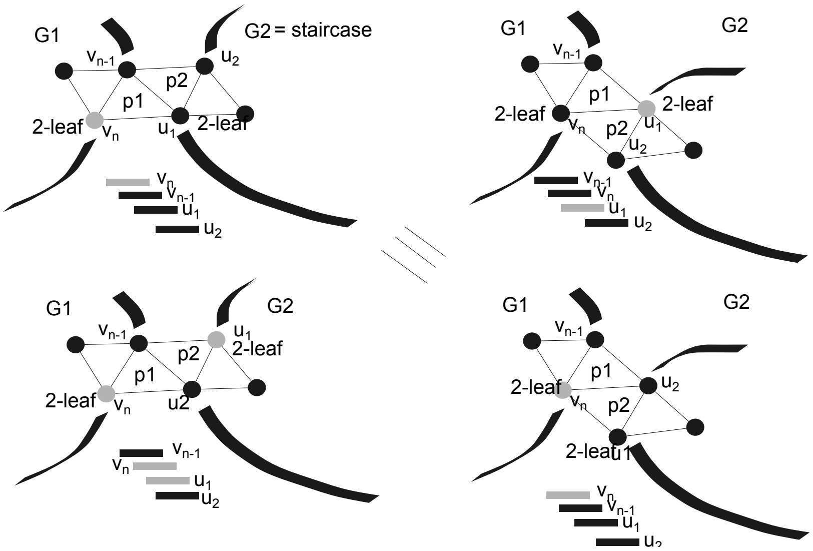
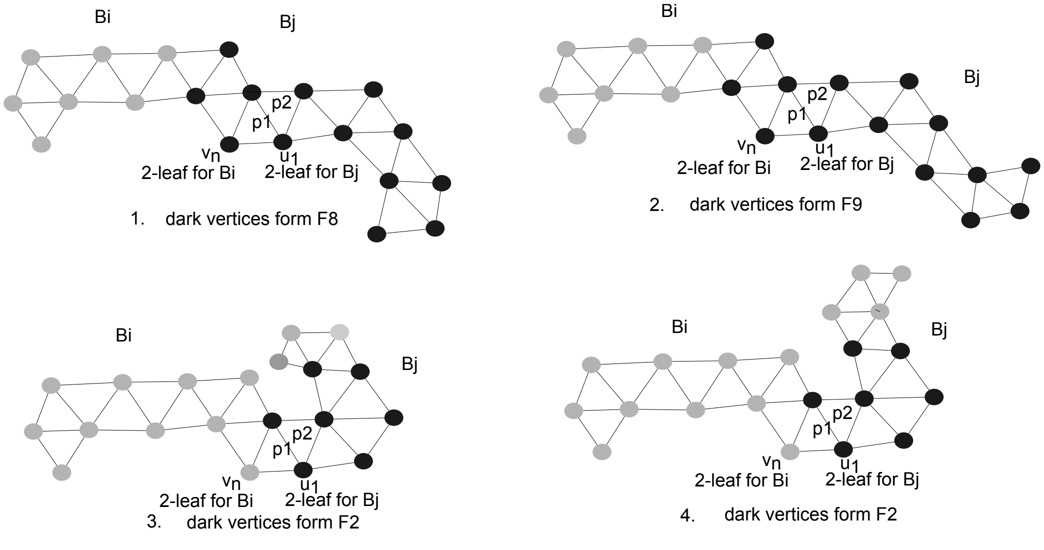


Figure 4.66: $n = 2$: vertex-consecutive-edge-consecutive for 2-path



1. dark vertices form F8
2. dark vertices form F9
3. dark vertices form F2
4. dark vertices form F2

Figure 4.67: $n = 2$: vertex-consecutive (2-snake)-edge-consecutive for 2-path

$F2, F3, F4, F5, F6, F7, F8, F9, F10, F11$ as an induced subgraph.

Lemma 4.5.25 *Let G be a 2-path and $G' = G-$ { end 2-leaves of G }. If G' is a straight 2-path, 2-snake or staircase, then G is a unit probe interval graph.*

Proof: Case1: Let G' be a straight 2-path. If we add back the end 2-leaves then we get two possible 2-paths. We either get a continuation of the straight 2-path (which has a unit probe interval representation) or a structure as shown in Figure 4.68. The unit probe interval representation is also given in the same figure.

Case2: Let G' be a staircase. If we add back the end 2-leaves then we either get a 5-fan or a continuation of the staircase (which has a unit probe interval representation) as shown in Figure 4.68. The unit probe interval representation is also given in the same figure.

Case3: Let G' be a 2-snake. If we add back the end 2-leaves then we either get a 4-fan or a continuation of the 2-snake as shown in Figure 4.70. The unit probe interval representation of both the structures are given in the same figure. ■

The following theorem will give a complete characterization of 2-paths that are unit probe interval graphs. In the next sections we will take the result one step further and figure out the complete list of forbidden subgraphs for interior 2-caterpillars which are just 2-paths with 2-leaves adjacent to some of interior edges. Since interior 2-caterpillars becomes a 2-path once the 2-leaves are removed, so the list of forbidden subgraphs for interior 2-caterpillars which have unit probe interval representation must also include the list of forbidden subgraphs for 2-paths which are unit probe interval graphs.

Theorem 4.5.26 *A 2-path is a unit probe interval graph if and only if it does not have F2, F3, F4, F5, F6, F7, F8, F9, F10, F11 as generated subgraphs.*

Proof: Let G be any 2-path. First we will prove the necessary part. Let us assume that G is a unit probe interval graph. Since $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$,

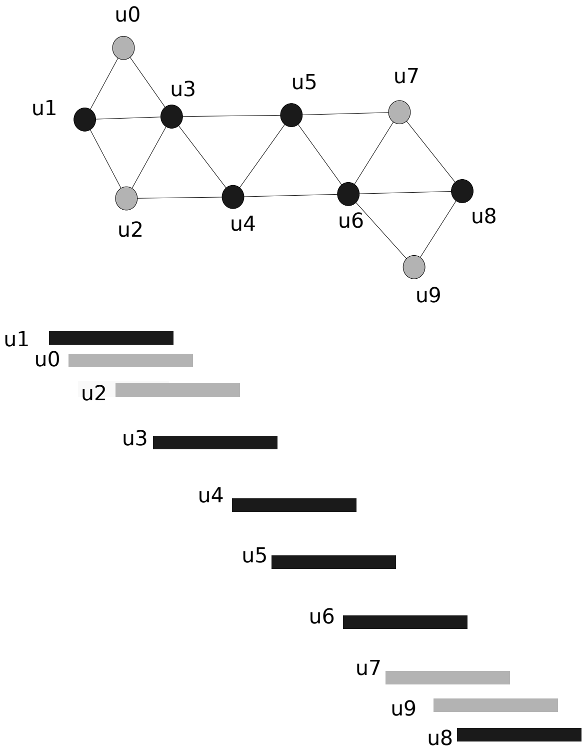


Figure 4.68: Straight-2-path+end-2-leaves

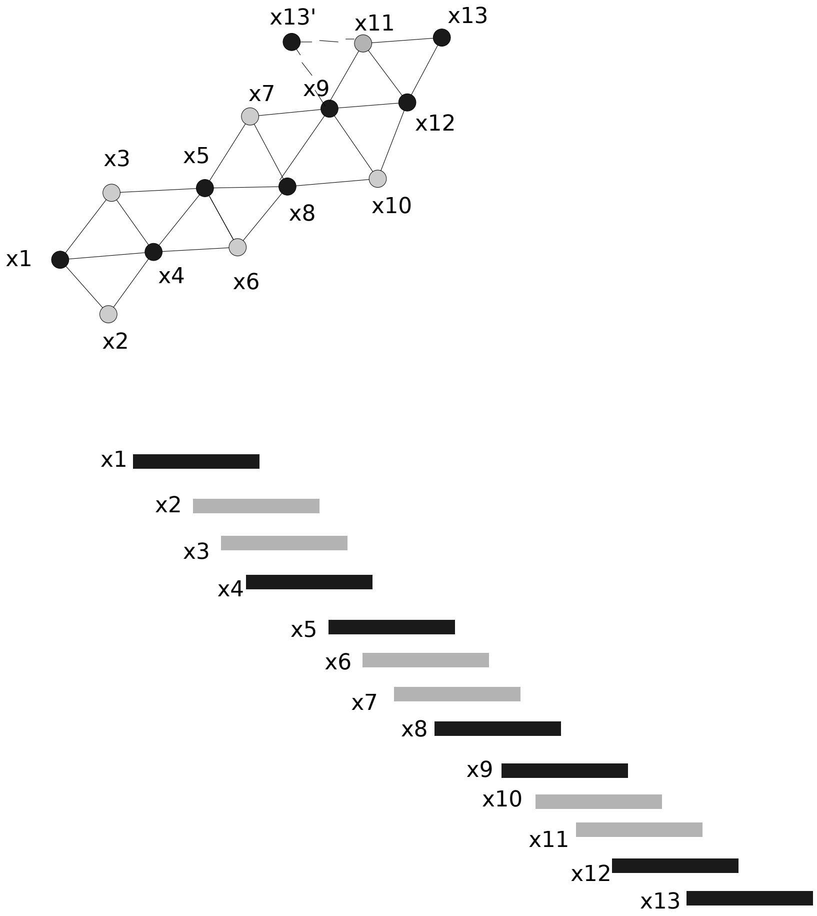


Figure 4.69: Staircase+end-2-leaves

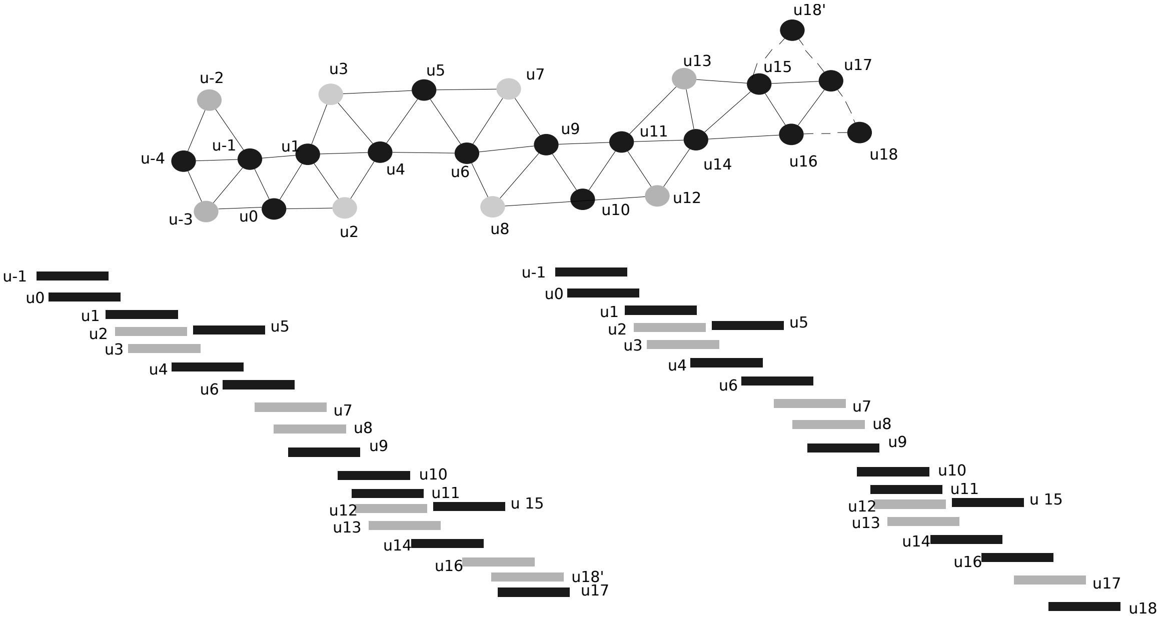


Figure 4.70: 2-snake+end-2-leaves

$F9$, $F10$, $F11$ do not have unit probe interval representations, any graph which has a unit probe interval interval representation cannot contain them as induced subgraphs. Thus any 2-path with an unit probe interval interval representation cannot contain them as induced subgraphs. So G does not have $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as generated subgraphs.

Now we will prove the sufficient part. Let us assume that G is 2-path without $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. Since G is a 2-path it can be considered as groups of edge or vertex consecutive 4-fans joined together by straight 2-paths. Since G does not contain any induced $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ by the previous theorems it must have a unit probe interval representation.

Thus it can be concluded that the complete list of forbidden subgraphs for any 2-path is given by $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$. ■

4.6 forbidden subgraphs for interior 2-caterpillars which are unit probe interval graphs

In this section we will provide a list of interior 2-caterpillars that do not have any unit probe interval representation. We call them $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$ and $E10$. We have already discussed the interior 2-caterpillar E1 in section 2 where we have proved that it does have any unit probe interval representation. As we have discussed earlier, 2-paths can be considered as the skeleton of interior 2-caterpillars. So if we are looking at interior 2-caterpillars which have unit probe interval representation we must first avoid those 2-paths that fail to be unit probe interval graphs. From the previous section we have the complete list of forbidden subgraphs of 2-path unit probe interval graphs which are $F2, F3, F4, F5, F6, F7, F8, F9, F10$ and $F11$. Furthermore we also know that since we are now dealing with structures that come from 2-paths by attaching 2-leaves to just the interior edges, then we must avoid $F1$ (which is not a 2-path) too. Hence right now, to achieve our goal, we must precisely look at interior 2-caterpillars whose underlying 2-paths (which itself is F1-free) does not contain $F2, F3, F4, F5, F6, F7, F8, F9, F10, F11$ as generated subgraphs and see which interior edges in these 2-paths on being attached to 2-leaves pose a problem for them to continue to be unit probe interval graphs. To find out exactly which interior edges cannot have 2-leaves attached, we will first load the 2-path unit probe interval graph with 2-leaves on every edge and try to find the corresponding unit probe interval representations. The moment we fail to get one we get our forbidden subgraph and hence the list consisting of Ei's is formed. We also have some other important results in this section (more precisely the following three results) which helps to locate the vertices in an interior 2-caterpillar which have probe, non-probe restrictions.

Lemma 4.6.1 *If there is 2-leaf on the first or last K_3 of a 4-fan then it must be a non-probe along with the end radial vertex that is located on the K_3 which has the*

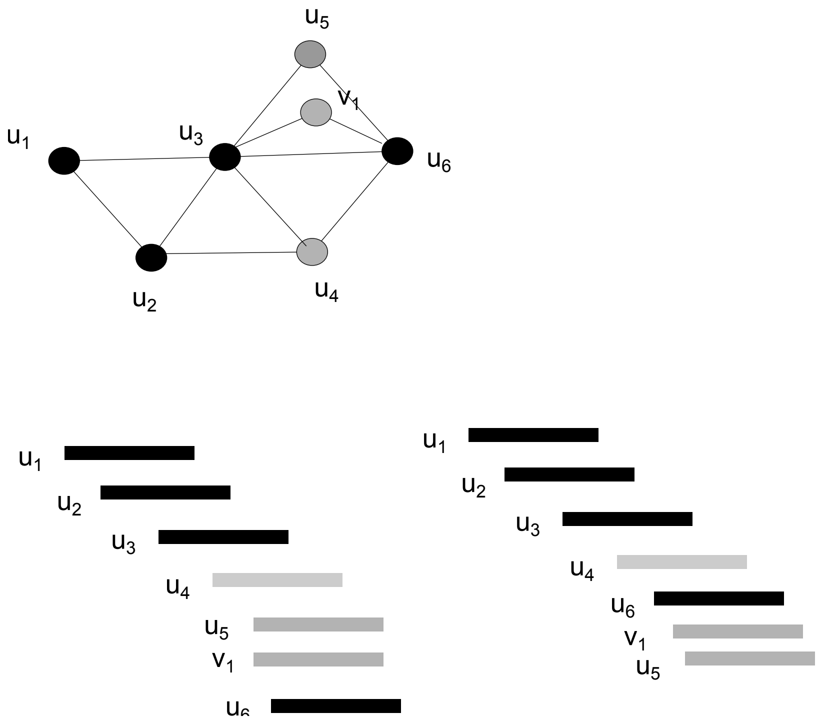


Figure 4.71: Non-probe restriction in 4-fans

2-leaf.

Proof: Let us consider the 4-fan with a 2-leaf v_1 on the interior edge u_3u_6 of the end K_3 given by $u_3u_6u_5$ as in figure 4.71. The central radial vertex u_4 must be a non-probe. So u_2 and u_6 must be probes. Vertices u_3, u_5, v_1, u_2 form an induced $K_{1,3}$ with the center at u_3. Since u_2 is a probe then u_5 and v_1 must be non-probes.

■

Lemma 4.6.2 *If G is a straight 2-path of length $n \geq 4$, denoted by $e_0, t_1, ...t_n, e_n$ that has a 2 leaf v_1 which is adjacent to an e_i such that $i \neq 0, 1, n-1, n$ (as in Figure 4.72) then v_1 and the vertices which are opposite to the edge e_i must be non-probes.*

Proof: We will prove the result by contradiction. Without loss of generality we can assume that n=4. Let G be a straight 2-path $e_0, t_1, ...t_4, e_4$ with a 2-leaf at e_2 as in Figure 4.72and assume that G is also a unit probe interval graph. let us assume

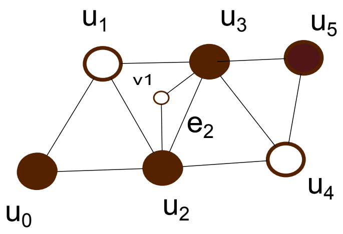


Figure 4.72: Non-probe restriction

that either u_1, v_1 or u_4 is a probe.

Case1: Suppose that v_1 is a probe. It can be easily seen from the figure that u_2, u_0, v_1, u_4 form an induced $K_{1,3}$ with the center at u_2. So u_2 must be a probe. Since v_1 is a probe, u_0 and u_4 must be non-probes. Furthermore, u_3, u_1, v_1, u_5 form another induced $K_{1,3}$ with center at u_3. So u_3 must be a probe. Similarly, since v_1 is a probe, u_1 and u_5 must be non-probes. Hence u_0 and u_1 both are non-probes but $u_0 \leftrightarrow u_1$ and hence we get a contradiction. So v_1 must be a non-probe.

The proofs for the cases that either u_1 or u_4 are probes is similar. Hence all 3 vertices must be non-probes. ■

It is easy to see that G is a unit probe interval graph with representation in Figure 4.73.

Lemma 4.6.3 *If there is 2-leaf on the middle* K_3 *of a 3-fan as shown in Figure 4.74 then, it must be a non-probe along with the end radial vertex which is located on the* K_3 *which shares the 2-leaf with the middle* K_3*.*

Proof: Let us consider the 3-fan with a 2-leaf v_1 on the interior edge u_2u_3 of the middle K_3 given by $u_2u_3u_4$ as shown in Figure 4.74. We will prove this by contradiction. Let us first assume that v_1 is a probe. It is observable that u_2, u_1, v_1, u_4

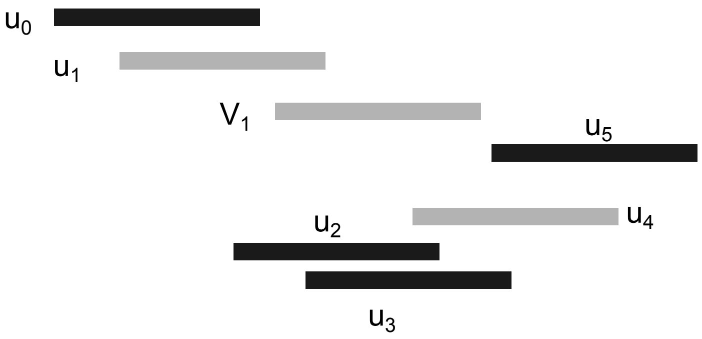


Figure 4.73: Unit probe interval representation

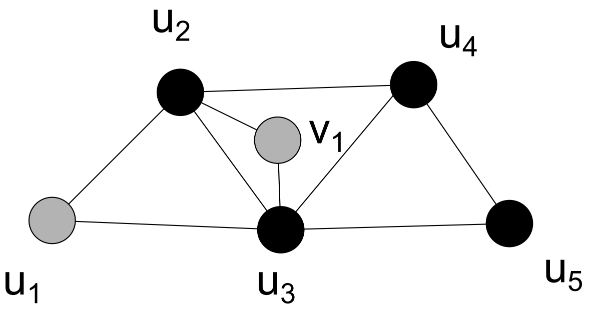


Figure 4.74: 3-fan with a 2-leaf

forms an induced $K_{1,3}$. Since v_1 is a probe then u_1 and u_4 must be non-probes. Furthermore, u_3, u_1, v_1, u_5 forms another induced $K_{1,3}$. Since v_1 is a probe so u_1 and u_5 must be non-probes. Hence both u_4 and u_5 are non-probes. But $u_4 \leftrightarrow u_5$. Hence we get a contradiction. Thus v_1 must be a non-probe.

Now we prove that either u_1, v_1, u_5 or u_1, v_1, u_4 must be non-probes. We already know that v_1 is a non-probe. Let us assume that u_1 is a probe which implies u_4, u_5 must be non-probes by the same argument used when we showed v_1 a non-probe. Hence we again get the same contradiction. So u_1 must also be a non-probe.

■

The next several lemmas establish the list of forbidden subgraphs.

Lemma 4.6.4 *The 2-tree probe interval graph which we will denote by E2 as in Figure 4.75 is not a unit probe interval graph.*

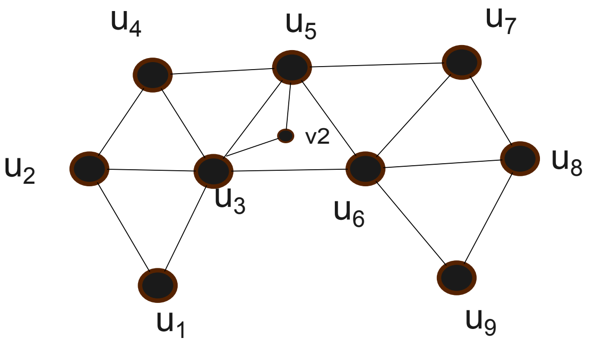


Figure 4.75: E2

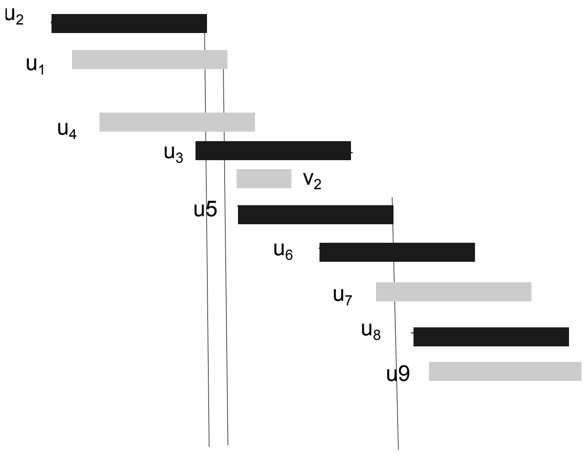


Figure 4.76: probe interval representation for E2

Proof: Let us assume that $E2$ has a unit probe interval representation. The vertices u_3 and u_6 are probes since they are the central vertices of 4-fans. Also u_4 and u_7 must be non-probes since they are the central radial vertices of the 4-fans with centers at u_3 and u_6. Hence u_2, u_5 and u_8 must be probes. Again, u_3, u_2, v_2, u_6 is an induced $K_{1,3}$ with the center at u_3. Hence either u_2 or u_6 must be a non-probe. But both u_2 and u_6 are probes which is a contradiction. Hence $E2$ is not a UPIG. ■

Note1: From the above two Lemmas it can be concluded that two edge connected 4-fans which represent a 2-snake and have unit probe interval representation can have 2-leaves only in the first or last K_3 of the 2-path otherwise we will get forbidden

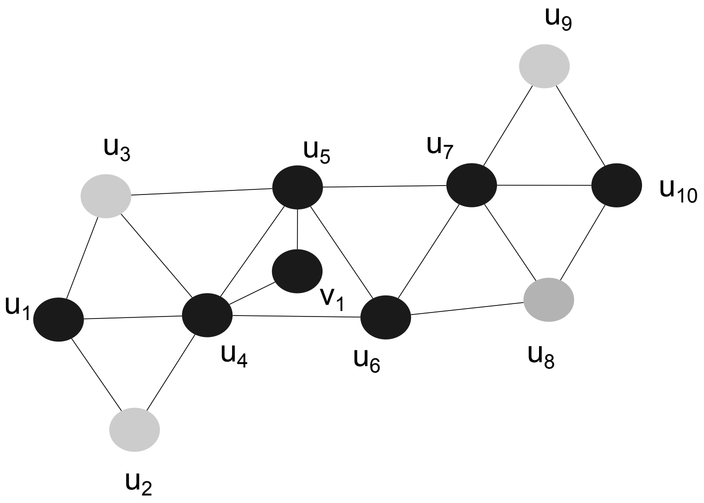


Figure 4.77: E3

subgraphs $E1, E2$.

Lemma 4.6.5 *The 2-tree probe interval graph which we will denote by E3 as in Figure 4.77 is not a unit probe interval graph.*

Proof: Let us assume that $E3$ as in Figure 4.77 has a unit probe interval representation. Vertices u_4 and u_7 are probes since they are centers of two 4-fans. Also u_3 and u_8 must be non-probes since they are the central radial vertices of the same 4-fans with centers at u_4 and u_7 respectively. Hence u_1, u_5, u_6, u_{10} must be probes. It is observable that u_4, u_1, v_1 and u_6 is an induced $K_{1,3}$ with center at u_4. Hence either u_1 or u_6 must be a non-probe. But both u_1 and u_6 are probes which is a contradiction. Hence $E3$ is not a UPIG. ■

Lemma 4.6.6 *The 2-tree probe interval graph which we will denote by E4 as in Figure 4.78 is not a unit probe interval graph.*

Proof: Let us assume that $E4$ as in Figure 4.78 has a unit probe interval representation. Vertices u_4 and u_7 are probes since they are centers of two 4-fans. Also vertices u_3 and u_8 must be non-probes since they are the central radial vertices

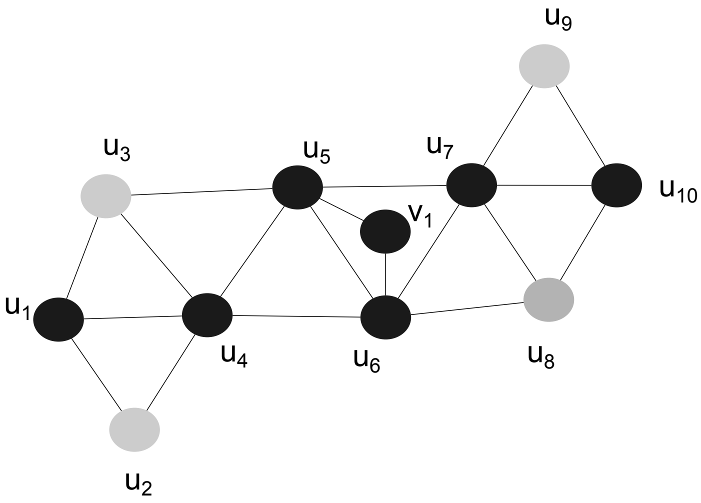


Figure 4.78: E4

of the same 4-fans with centers at u_4 and u_7 respectively. Also u_5, u_4, v_1 and u_7 is an induced $K_{1,3}$ with center at u_5. Hence either u_4 or u_7 must be a non-probe. But both u_4 and u_7 are centers of 4-fans and hence they must be probes which is a contradiction. Hence $E4$ is not a Unit probe interval graph. ■

Note2: From the above two lemmas it can be concluded that two edge connected 4-fans which represent an extended-staircase and have unit probe interval representation can have 2-leaves only in the first or last K_3 of the 2-path otherwise we will get forbidden subgraphs $E1$, $E3$, $E4$.

Lemma 4.6.7 *The 2-tree probe interval graph which we will denote by E5 as in Figure 4.79 is not a unit probe interval graph.*

Proof: Let us assume that $E5$ has a unit probe interval representation. By Lemma 4.6.2, u_1, v_1 and u_4 must be non-probes. Hence u_3 must be a probe. Also u_5, u_3, v_2 and u_6 form an induced $K_{1,3}$ with u_5 as the center vertex. Since u_3 is a probe So v_2 and u_6 must be non-probes. Hence u_6 and u_4 are both non-probes even though $u_6 \leftrightarrow u_4$ which is a contradiction. So $E5$ cannot have a unit probe interval representation. ■

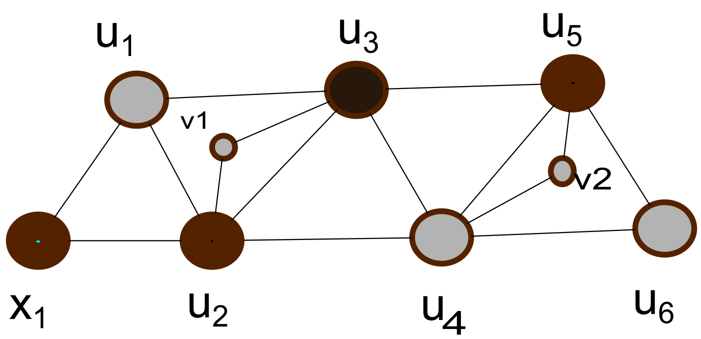


Figure 4.79: E5

The probe interval representation of E5 is given in Figure 4.80.

Lemma 4.6.8 *The 2-tree which we will denote by E6 as in Figure 4.81 is not a unit probe interval graph.*

Proof: Let us assume that $E6$ has a unit probe interval representation. By Lemma 4.6.2 u_1, v_1 and u_4 are non-probes. Thus v_2 must be a probe. Since u_2, u_5

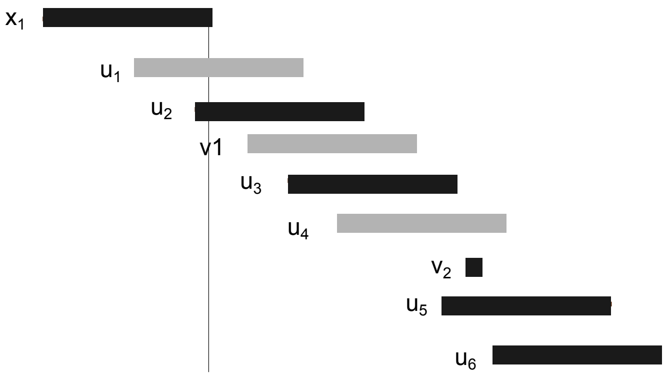


Figure 4.80: Probe interval representation of E5

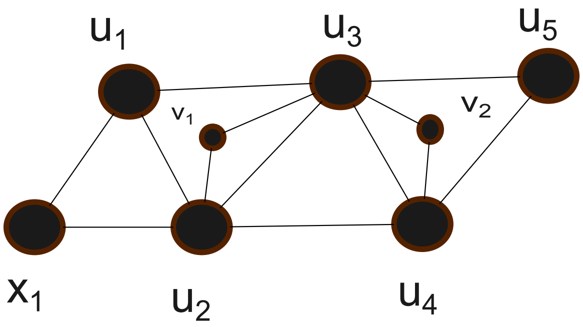


Figure 4.81: E6

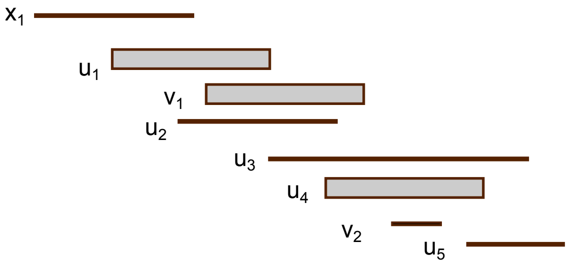


Figure 4.82: Probe interval representation of E6

and v_2 are the 3 ends of a K_{13} with center at u_3, then u_2 and u_5 must be non-probes. Since u_4 and u_5 are adjacent and both are non-probes, we get a contradiction. Hence $E6$ is not a UPIG.

■

Lemma 4.6.9 *The 2-tree probe interval graph which we will denote by E7 as in Figure 4.83 is not a unit probe interval graph.*

Proof: Let us assume that $E7$ as in Figure 4.83 has a unit probe interval representation. Vertices u_4 and u_8 are probes since they are centers of two 4-fans. Further u_6, u_4, v_1, u_8 is an induced $K_{1,3}$ with center at u_6. Hence either u_4 or u_8 or both must be a non-probe. But both u_4 and u_8 are centers of 4-fans and hence they must be probes which is a contradiction. Hence $E7$ is not a UPIG. ■

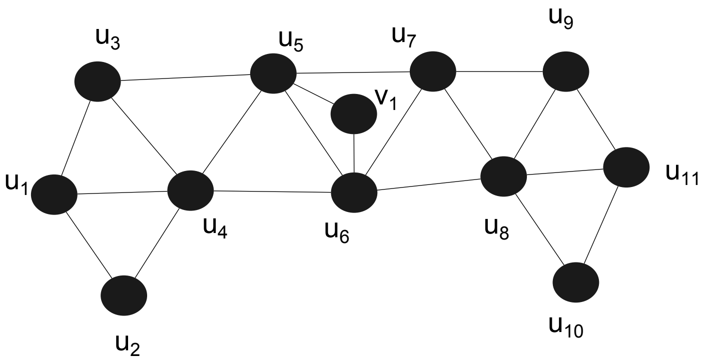


Figure 4.83: E7

Note3: From the above Lemma it can be concluded that if two 4-fans are joined by a straight 2-path of length one then that straight 2-path cannot have any 2-leaf on any of its interior edge otherwise we will get the forbidden subgraph $E7$. In other words two vertex-consecutive 4-fans Fan1 and Fan2 cannot have any 2-leaf on the interior edges of the K_3 which contains the common vertex shared by both Fan1 and Fan2.

Lemma 4.6.10 *The 2-tree probe interval graph which we will denote by E8 as in Figure 4.84 is not a unit probe interval graph.*

Proof: Let us assume that $E8$ as in Figure 4.84 has a unit probe interval representation. Vertices u_3 and u_8 are centers of two 4-fans with 2-leaves v_1 and v_2 on their end K_3's u_3, u_6, u_5 and u_7, u_8, u_9 respectively. So by Lemma 4.6.1 u_5 and u_7 must be non-probes. But $u_5 \leftrightarrow u_7$ which is a contradiction. Hence $E8$ is not a UPIG.

■

Lemma 4.6.11 *The 2-tree probe interval graph which we will denote by E9 as in Figure 4.85 is not a unit probe interval graph.*

Proof: Let us assume that $E9$ as in Figure 4.85 has a unit probe interval representation. The vertices u_4 and u_{10} are centers of two 4-fans with 2-leaves v_1 and

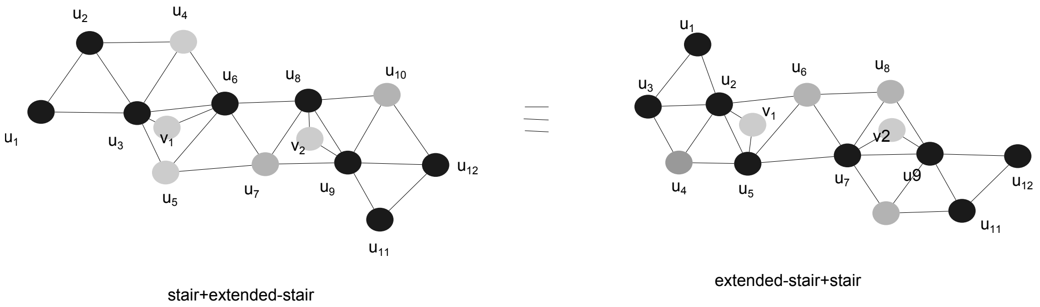


Figure 4.84: E8

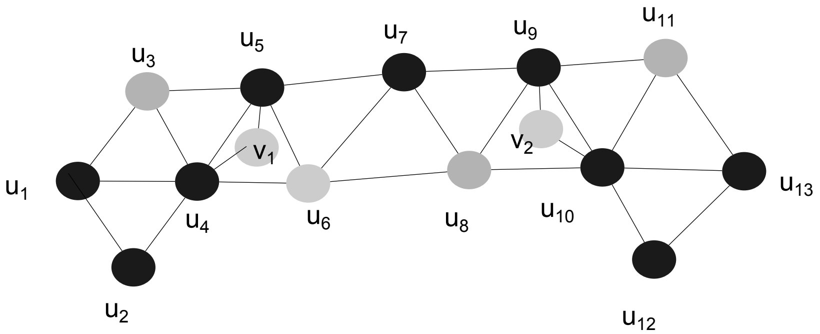


Figure 4.85: E9

v_2 on their end K_3's u_4, u_6, u_5 and u_{10}, u_8, u_9 respectively. So by 4.6.1 u_6 and u_8 must be non-probes. But $u_6 \leftrightarrow u_8$ which is a contradiction. Hence $E9$ is not a UPIG. ■

Lemma 4.6.12 *The 2-tree probe interval graph which we will denote by E10 as in Figure 4.86 is not a unit probe interval graph.*

Proof: Let us assume that $E10$ as in Figure 4.86 has a unit probe interval representation. The vertices u_2 and u_9 are centers of two 4-fans with 2-leaves v_1 and v_2 on their end K_3s u_2, u_6, u_5 and u_7, u_8, u_9 respectively. So by Lemma 4.6.1 u_6 and u_7 must be non-probes. But $u_6 \leftrightarrow u_7$ which is a contradiction. Hence $E10$ is not a UPIG. ■

Lemma 4.6.13 *The 2-trees which we will denote by Group-*$E11$ *as in Figure 4.87 is not a unit probe interval graph.*

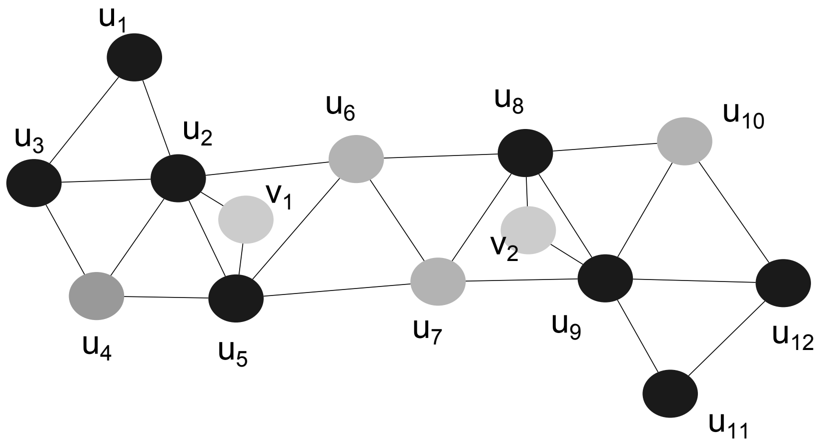


Figure 4.86: E10

Proof: Let us assume that the graphs in Figure 4.87 have a unit probe interval representation. We know that u_1 is a non-probe and u_2 which is the left end 2-leaf of 2-snake or extended-staircase is also a non-probe. But $u_1 \leftrightarrow u_2$ which is a contradiction. ■

1

4.7 Characterization of interior 2-caterpillars which are unit probe interval graphs

This is the final section of our chapter. In this section we characterize 2-tree that are unit probe interval graphs. In section 3 we proved that 2-tree unit probe interval graphs are interior 2-caterpillars. Our main goal was to characterize 2-tree unit probe interval graphs. So this result from section 3 reduced our range of graphs drastically and we narrowed our concerns to just interior 2-caterpillars which are just 2-caterpillars without an $F1$ or in other words a 2-path with 2-leaves only on the interior edges. First we just looked at 2-paths and then added the 2-leaves to finally get the characterization of interior 2-caterpillar which are unit probe interval graphs. Using this and the stated result from section 3 we prove in this section that 2-trees are unit probe interval graphs if and only if they are interior 2-caterpillars without Fi's i=1,2,......11, Ej's j=1, 2,....10 and Group-E11. Finally we have the proof of the

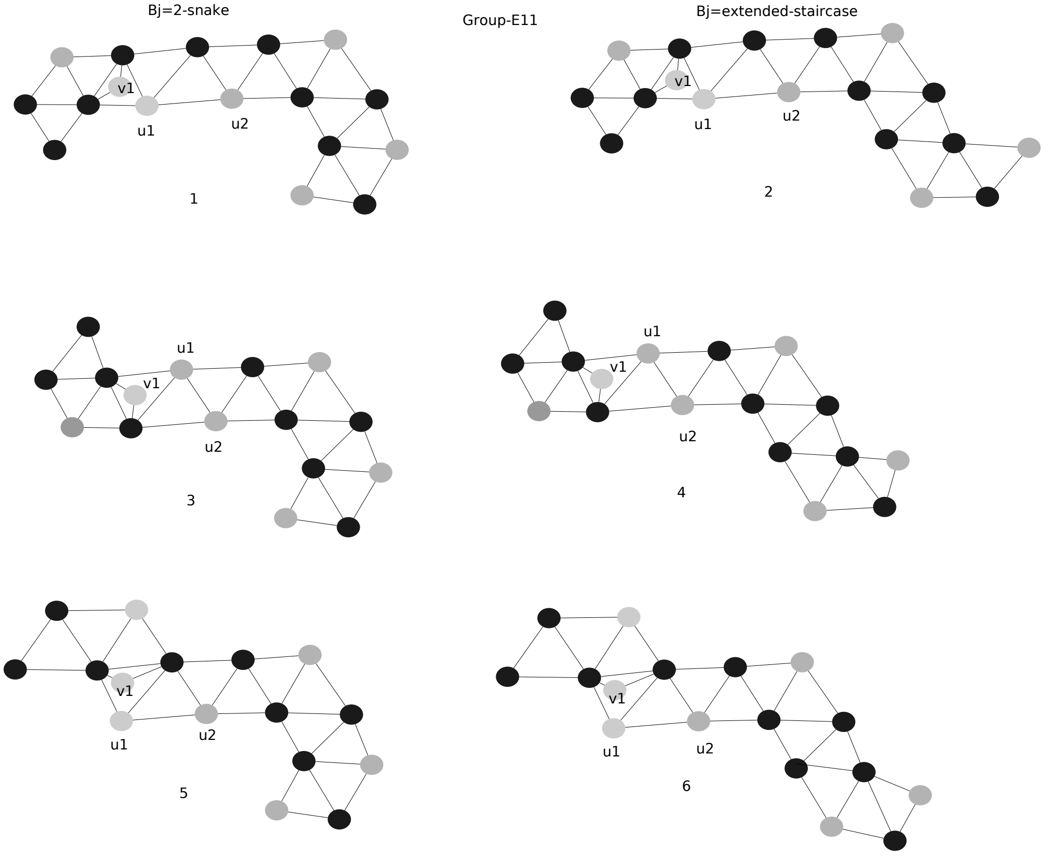


Figure 4.87: Group-E11- u1,u2 are end 2-leaves of B_i and B_j

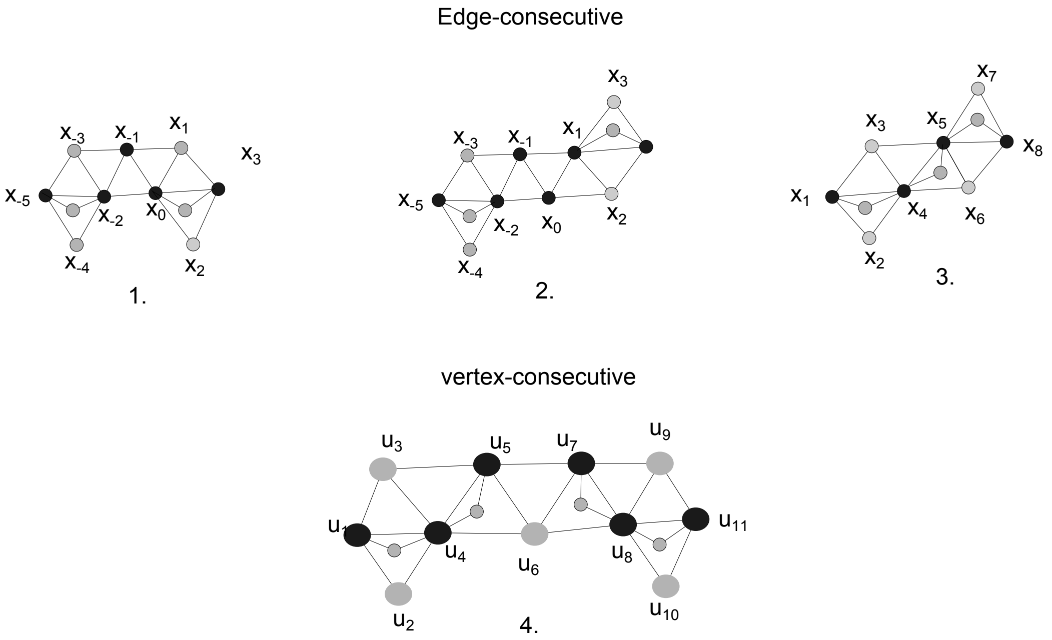


Figure 4.88: Possible places of 2-leaves

characterization of 2-tree unit probe interval graphs.

From Note1, Note2 and Note3 of the previous section we can conclude that two edge-consecutive or vertex-consecutive 4-fans which are also unit probe interval graphs can possibly have 2-leaves only in the following manner as illustrated in Figure 4.88. The unit probe interval representation of pictures 1, 2, 3, 4 of the same Figure 4.88 are given in Figures 4.89, 4.93, 4.90, 4.94 4.91, 4.92. So a 2-snake with all possible 2-leaves, a staircase with all possible 2-leaves and an extended staircase with all possible 2-leaves (it must be noted here that an extended staircase beyond a merge of two 4-fans gives a forbidden subgraph for a 2-path and so we do not consider it, instead we look at it as a merge with a staircase or 2-snake) have a unit probe interval representation.

let us discuss some important points about a 2-path. We will construct a 2-path from the beginning. We start with a 2-complete graph, K_2, which is a 2-tree and call it G. We keep adding new vertices adjacent to a 2-complete subgraph of G such that

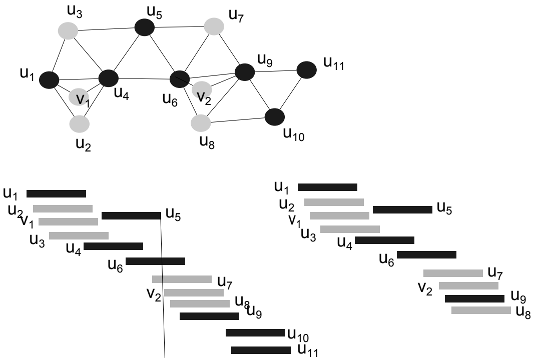


Figure 4.89: Representation of 2-snake with all possible 2-leaves

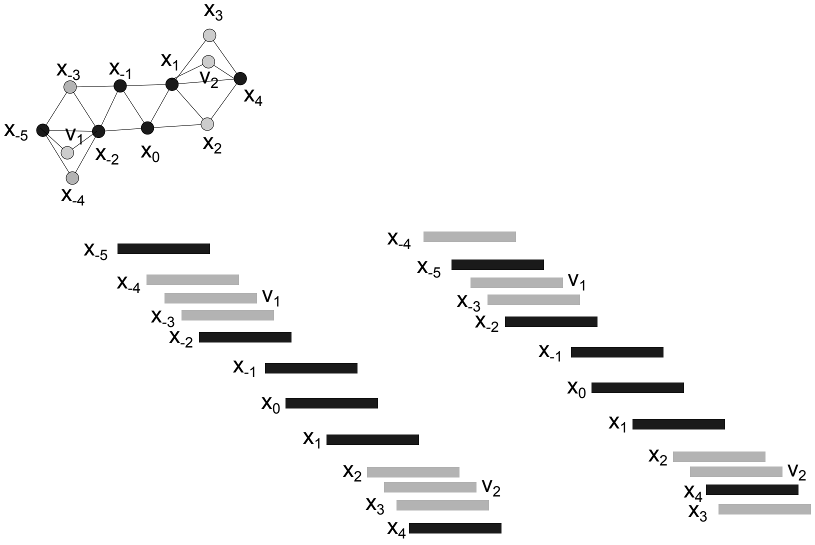


Figure 4.90: Representation of extended staircase with all 2-leaves

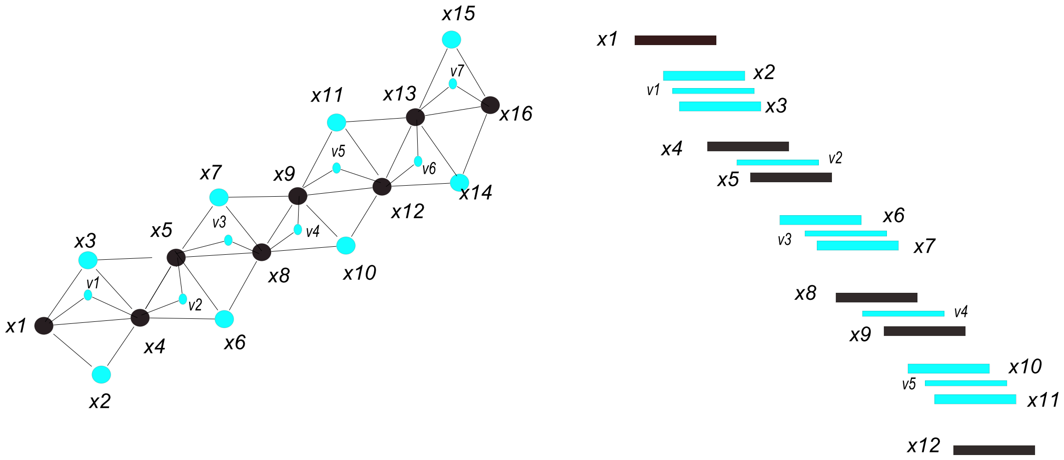


Figure 4.91: Representation of staircase with all 2-leaves

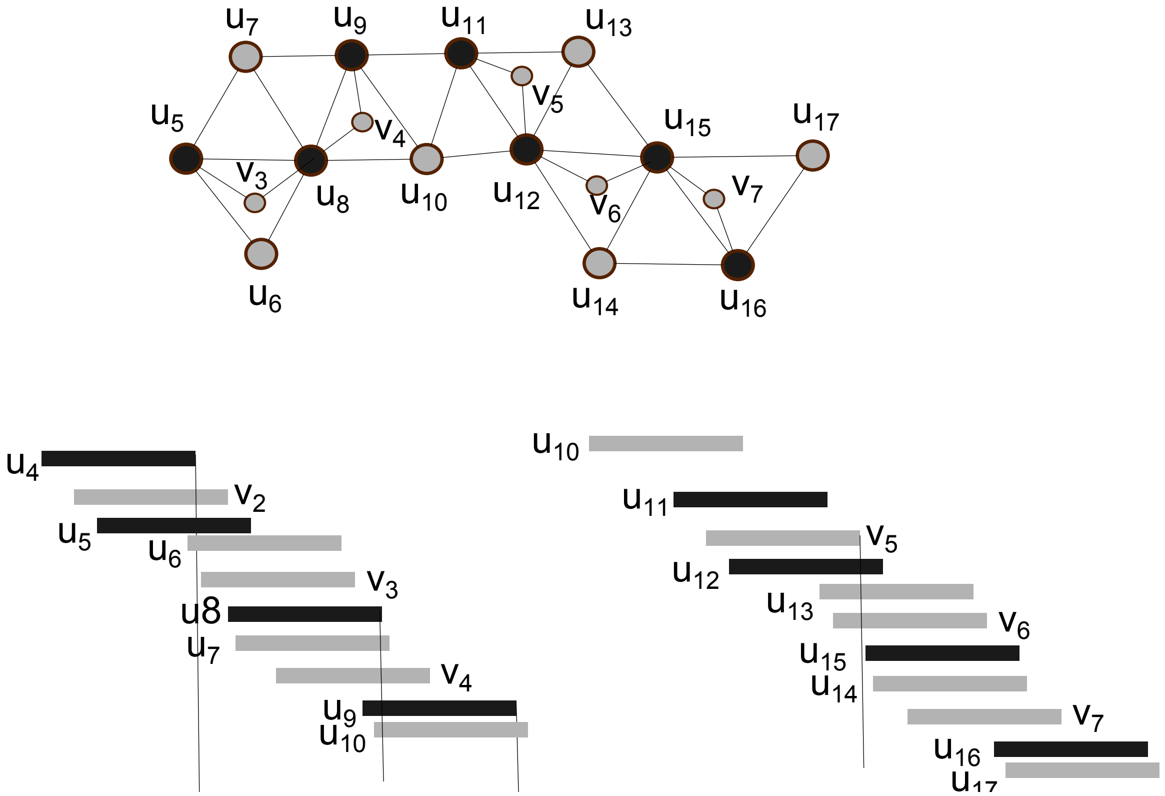


Figure 4.92: Representation of vertex-consecutive 4-fans with all 2-leaves

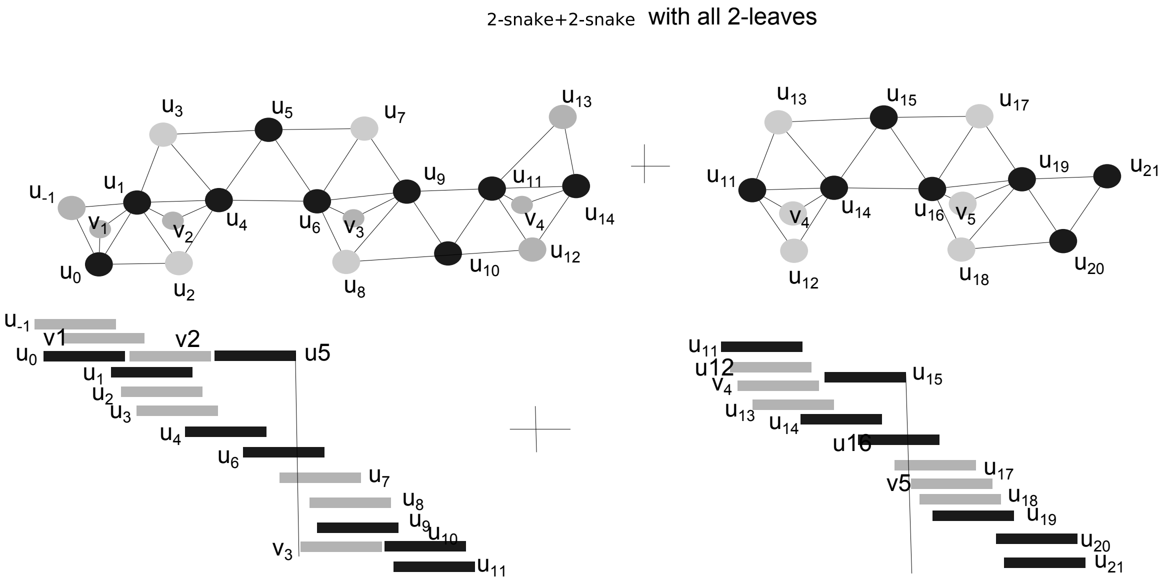


Figure 4.93: Representation of 2-snake with all 2-leaves

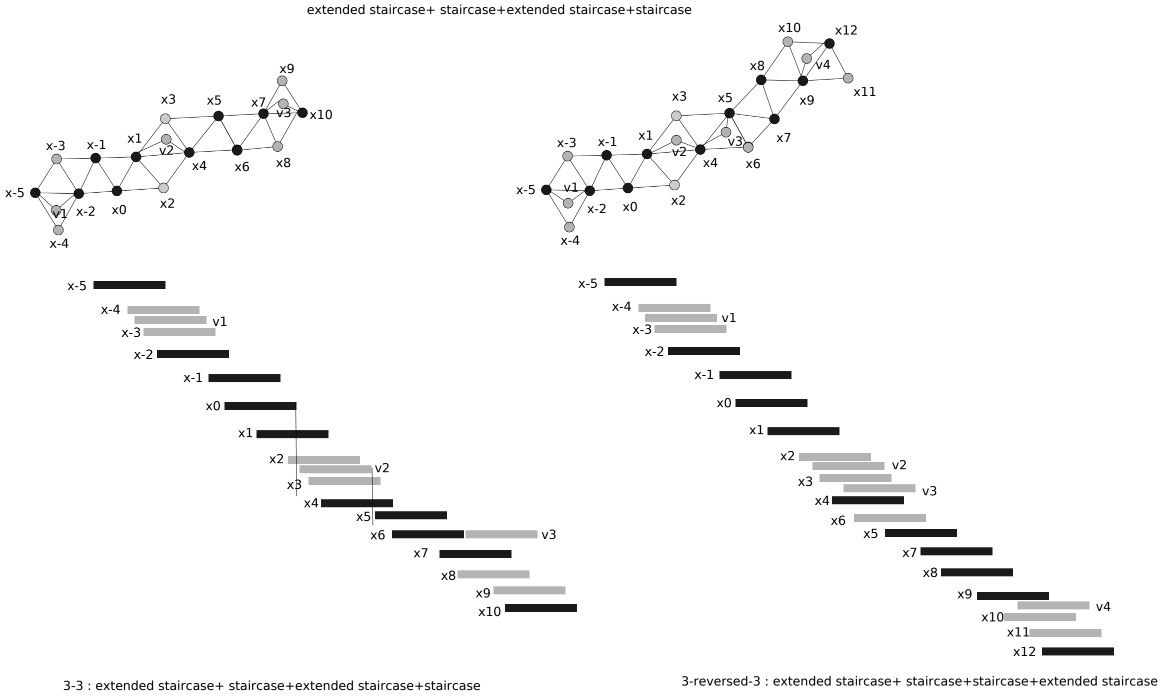

extended staircase+ staircase+extended staircase+staircase
x-5
x-4
x-3
x-2
x-1
x0
x1
x2
x3
x4
x5
x6
x7
x8
x9
x10
v1
v2
v3
x-5
x-4
v1
x-3
x-2
x-1
x0
x1
x2
v2
x3
x4
x5
x6
v3
x7
x8
x9
x10
3-3 : extended staircase+ staircase+extended staircase+staircase
x-5
x-4
x-3
x-2
x-1
x0
x1
x2
x3
x4
x5
x6
x7
x8
x9
x10
x11
x12
v1
v2
v3
v4
x-5
x-4
v1
x-3
x-2
x-1
x0
x1
x2
v2
x3
v3
x4
x6
x5
x7
x8
x9
v4
x10
x11
x12
3-reversed-3 : extended staircase+ staircase+staircase+extended staircase

Figure 4.94: Representation of extended-staircase and staircase with all 2-leaves

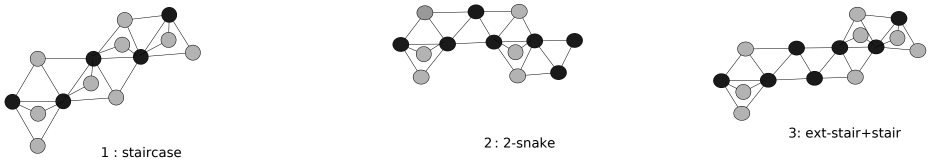


Figure 4.95: Edge-consecutive 4-fans with all 2-leaves

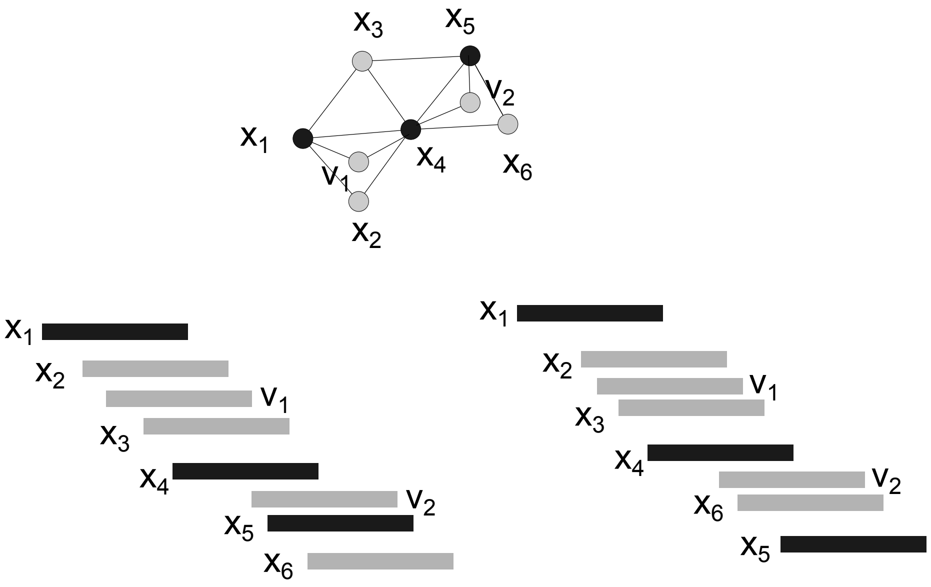


Figure 4.96: 4-fan with all 2-leaves

the new vertex is either adjacent to the two most recently added vertices or not. If it is adjacent to the two most recently added vertices we get a 2-tree which is a straight 2-path of 2 and 3 cliques. We call it a straight 2-path.

If not then we either get an $F1$ or a 4-fan. After avoiding an $F1$ (since we are building a 2-path) if a new vertex is added and it is adjacent to exactly one of the 2 most recently added vertices, then it creates a 4-fan.

Hence it can be concluded from the discussion so far that with the addition of a vertex to an existing 2-path either a 4-fan is created or not. If not then we get a straight 2-path. We also know that when a 2-path is a merge of more than one 4-fan it can be either: (1) an edge-consecutive or (2) a vertex-consecutive. Keeping these in mind, 2-paths can be categorized into 3 groups in a generalized way as follows:

1. Z_1: 2-paths which are formed by groups of edge-consecutive 4-fans joined by straight 2-paths.
2. Z_2: 2-paths which are formed by groups of vertex-consecutive 4-fans joined by straight 2-paths .
3. Z_3: 2-paths which are formed by a group of edge-consecutive 4-fans joined by straight 2-paths to a group of vertex-consecutive 4-fans.

From these definitions we derive three categories of interior 2-caterpillars as follows:

1) I_1: Interior 2-caterpillars which are formed by attaching the interior edges of Z_1 with all possible 2-leaves.

2) I_2: Interior 2-caterpillars which are formed by attaching the interior edges of Z_2 with all possible 2-leaves.

3) I_3: Interior 2-caterpillars which are formed by attaching the interior edges of Z_3 with all possible 2-leaves.

From the previous results on the characterization of 2-paths we know that there exists 3 possible structures for two edge-consecutive 4-fans and one possible structure for two vertex-consecutive 4-fans. In the case of edge-consecutive they have specific names as given in Figure 4.27:-

a) staircase

b) 2-snake

c) extended staircase

The vertex consecutive structure is given by Figure 4.95. We also know that both the 2-leaves of a 2-snake and extended-staircase have a non-probe restriction (by lemma 4.5.13). Furthermore any 2-leaf on an interior edge of the first or last K_3 of a 4-fan must also be a non-probe. From the unit probe interval representation of all the three possible two edge-consecutive 4-fans and one possible two vertex-consecutive 4-fans with all the possible 2-leaves, we can conclude that the intervals of 2-leaves can be added to them without disturbing the interval layout of the underlying 2-path such that the resultant graph is also a UPIG. Furthermore it is also worth noting here that $E1, E2, E3, E4, E7, E8$, Group-$E11$ are the only forbidden subgraphs whose underlying 2-paths are either edge or vertex consecutive 4-fans. The rest are related to straight 2-paths. We will use these facts to prove the following lemmas.

Lemma 4.7.1 *Any number of edge consecutive 4-fans with all its possible 2-leaves (that forms a 2-path without the 2-leaves) has a unit probe interval representation if and only if there is no induced F2, F3, F4, F5, F6, F7, F8, F9, F10 and F11 and E1, E2, E3, E4, E5, E6 and E8 in it.*

Proof: First we prove the necessary part. Let G be n edge consecutive 4-fans $f_1, f_2, ...f_n$ with all possible 2-leaves. If G is unit a probe interval graph then G cannot contain $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ and $E1, E2, E3, E4, E5, E6$ and $E8$ as induced subgraphs.

Now we prove the sufficient part. Let G be n edge consecutive 4-fans $f_1, f_2, ...f_n$ with all possible 2-leaves and no induced $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$, $E1$, $E2$, $E3$, $E4$, $E5$, $E6$ and $E8$ in it. We will try to get a unit probe interval representation of G. It can be observed that out of all the forbidden subgraphs with 2-leaves only $E1, E2, E3, E4, E5, E6$ and $E8$ can occur in the case of edge-consecutive 4-fans and so we avoid just these subgraphs apart from all the Fi's to avoid any problem towards the representation of G. Let G'= G-(2-leaves). Since G' is a 2-path without $F2, F3, F4, F5, F6, F7, F8, F9, F10$ and $F11$, so G' has a unit probe interval representation. Now we look at G' as merge of two edge-consecutive 4-fans $f_1f_2, f_2f_3, ...f_if_{i+1}, ...$ From Figures 4.89, 4.90, 4.91 (which gives different possible merge of two edge-consecutive 4-fans with all possible 2-leaves such that no Ei's are formed as induced subgraphs in it), it can be concluded that the intervals of 2-leaves can be added to each of these blocks without disturbing the interval layout of the underlying 2-path such that the resultant graph is also a UPIG and the 2-leaves does not have any distinct end point. So no 2-leaf-interval from block f_{i-1} will overlap with intervals from f_{i+1}. Moreover all the 2-leaves are non-probes so that their overlapping with each other never matters. We now start adding the intervals of 2-leaves to the representation of G'. Furthermore, since G cannot have $E1, E2, E3, E4, E7, E8$ there are fixed places where the 2-leaves get attached to G'. Hence adding the 2-leaves of f_1f_2 first and inserting the new intervals corresponding to the 2-leaves in the representation of G' does not get affected by the successive addition of 2-leaf intervals for f_2f_3 (no 2-leaf interval from f_3 overlaps with intervals from f_1, 2-leaf interval additions also does initiate any change in the representation of the underlying 2-path and overlapping 2-leaves does not affect each others adjacency since they are all non-probes). The addition of intervals of 2-leaves is done, keeping in mind that no $E1, E2, E3, E4, E7, E8$ is formed in the transition of G' to G, till the whole graph has been covered. Thus we get a unit probe interval representation of G. ■

We now look at two vertex consecutive 4-fans (which is a unit probe interval graph from the previous section) with all possible 2-leaves and figure out the interior edges where we can add 2-leaves without disturbing the unit probe interval representation of the 2-path.

Theorem 4.7.2 *Two vertex-consecutive 4-fans with all possible 2-leaves without E1, E5, E6, E7 has a unit probe interval representation as shown in Figure 4.97.*

Proof:

Let G be two vertex-consecutive 4-fans with maximum possible 2-leaves as in Figure 4.97. We will give a unit probe interval representation of G.

We know that the middle edges u_7u_8 and $u_{12}u_{13}$ of the 4-fans cannot have 2-leaves otherwise we will get an $E1$. Furthermore on any part of the body of G which has at least six consecutive K_3s, if edge e_k has a 2-leaf then its next nearest 2-leaf must be on the edge e_{k+3} otherwise we will get the forbidden subgraphs $E5, E6$. Also it cannot have any 2-leaf on the interior edge of the K_3 that merges the two 4-fans otherwise we will get an induced $E7$.

Unit probe interval representation of G:

Let all the 2-leaves be non-probes. If the 2-leaf v_i is adjacent to the edge e_i then the vertices opposite to the edge e_i in the K_3's t_i and t_{i+1} are also non-probes. We will consider the vertices from $u_5 - u_{10}$ in G. Let us call this 4-fan f_i. Thus f_i has $u_5, u_6, u_7, u_8, u_9, u_{10}, v_3$ and v_4 in it. We will give the unit probe representation of the vertices of f_i and using the the same method we can get the unit probe representations of the next 4-fan $u_{10} - u_{15}$ called f_j and finally we will merge these two representations to get the representation of G. We will first draw the intervals corresponding to the non-2-leaves which are denoted by u_is as in Figure 4.97.

In f_i:

$r(I_{u_5}) < r(I_{u_6})$

$r(I_{u_6}) < r(I_{u_7})$

$l(I_{u_7}) \cap r(I_{u_5}) = \epsilon$

$l(I_{u_8}) \cap r(I_{u_5}) > \epsilon$

All these are illustrated in the Figure 4.97

In the f_j:

$r(I_{u_8}) < r(I_{u_9})$ such that

$I_{u_7} \cap I_{u_8} \cap I_{u_9} \neq \phi$, $r(I_{u_8}) < r(I_{u_7}) < r(I_{u_9})$ and $r(I_{u_8}) \cap l(I_{u_9}) \neq \phi$.

$r(I_{u_8}) < r(I_{u_9}) < r(I_{u_{10}})$ such that $l(I_{u_8}) \cap r(I_{u_{10}}) = \epsilon$.

Note that in this representation $I_{u_{10}} \cap I_{u_7} \neq \phi$ but it is ok since both of them are non-probes.

All these are illustrated in Figure 4.97

Now we figure out the position of the 2-leaves in the representation. It is easily seen from the figure that the 2-leaves are not attached to the middle edge of the 4-fans. Two 2-leaves are adjacent to two internal, non-middle, non-side edges of the 4-fans. Furthermore along any straight 2-path part of G a 2-leaf skips exactly 2 edges from its previous leaf. The intervals of 2-leaves are added to the present 2-path as in Figure 4.97.

Hence we get a unit probe interval representation of f_i. Now f_j starts from $u_{10} - u_{15}$. It is easy to see that f_j is isomorphic to f_i. So we do the same representation for f_j and merge it with f_i in the following way.

The last interval of f_i becomes the first interval of f_j. Since f_i ends with u_{10} such that$r(I_{u_9}) < r(I_{u_{10}})$, so ($I_{u_{10}}$ has a distinct right end point and we have enough space free space of the interval of u_{10} after u_9 ends so that we can make the intervals of u_{12}, v_5 overlap with u_{10} without overlapping with the interval of u_9.

■

Lemma 4.7.3 *Any number of vertex-consecutive 4-fans with all its possible 2-leaves (that forms a 2-path without the 2-leaves) has a unit probe interval representation if and only if there is no induced F2, E1, E5, E6, E7 in it.*

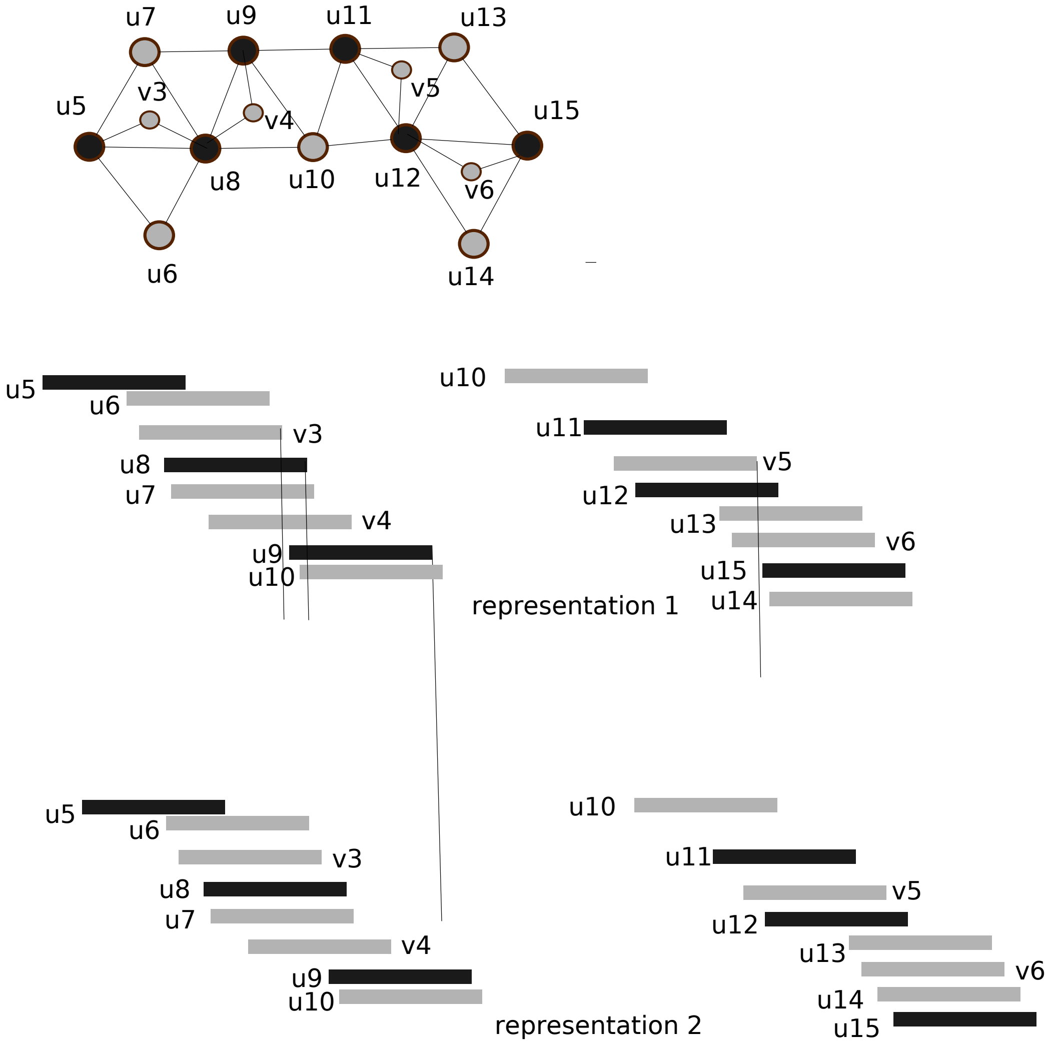


Figure 4.97: Vertex consecutive 4-fans with all 2-leaves

Proof: Let G be n vertex consecutive 4-fans $f_1, f_2, ...f_n$ with all possible 2-leaves. If G has unit probe interval representation then G must not contain $F2, E1, E5, E6, E7$ as induced subgraphs in it.

Now we prove the other direction. Let G be n vertex consecutive 4-fans $f_1, f_2, ...f_n$ with all possible 2-leaves. G does not contain any $F2$, $E1$, $E5$, $E6$, $E7$ as induced subgraphs in it. We will prove that G is a unit probe interval graph. Let G'= G-(2-leaves). G' is a 2-path made of vertex-consecutive 4-fans without $F2$ and so G' has a unit probe interval representation. Now we look at G' as merge of two vertex-consecutive 4-fans $f_1f_2, f_2f_3, ...f_if_{i+1}, ...$ From Figure 4.97, it can be concluded that the intervals of 2-leaves can be added to each of these blocks without disturbing the interval layout of the underlying 2-path such that the resultant graph is also a UPIG. It is also easy to see that out of all the forbidden subgraphs with 2-leaves only $E1, E5, E6, E7$ can occur in the case of vertex-consecutive 4-fans and so we avoid just them apart from the $F2$ in the underlying structure. Furthermore, it is also easy to see from the same figure that no 2-leaf from any interior edge has distinct left or right end point and so it can be concluded that the interior 2-leaves, when added to these blocks, does not overlap with any interval outside the block. Hence no 2-leaf interval from f_jf_{j+1} overlaps with any interval from f_{j-1} and f_{j+1}. Also, we can conclude from the same figure, that the removal of intervals corresponding to the 2-leaves from interior edges (such that we obtain 2-paths which are vertex-consecutive 4-fans) of any two vertex-consecutive 4-fans f_if_{i+1} does not affect the unit probe interval representation of the underlying 2-path of the block. So after the 2-leaves are added to G', it cannot affect the representation of any f_if_{i+1}. Since any number of vertex consecutive 4-fans which form a 2-path is a unit probe interval graph, we first do the unit probe representation of G'. Since the addition of 2-leaves on any two vertex-consecutive 4-fans f_if_{i+1} is independent of the unit probe representation of the underlying 2-path and the rest of the blocks, we can easily add the intervals corresponding to the interior 2-leaves to the representation of G' and finally get a unit probe interval representation of G. ■

We will now take each of I_1, I_2, I_3 into consideration such that nowhere the forbidden subgraphs $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ are formed as induced subgraphs and give a unit probe interval representation for each of them.

Theorem 4.7.4 *Interior* 2*-caterpillars given by structures isomorphic to merge of* I_1 *have unit probe interval representation if and only if they do not have E1, E2, E3, E4, E5, E6, E7, E8, E9, E9, E10, F2, F3, F4, F5, F6, F7, F3, F8, F9, F10, F11 as induced subgraphs.*

Proof: Let $B_1, B_2, B_3, ...B_m$ be m ($m \geq 2$) groups of edge-consecutive 4-fans with all possible 2-leaves and they are joined by straight 2-paths of length n ($n = 0, 1, 2....$) with all possible 2-leaves. This structure is isomorphic to merge of I_1 and we will call it G. If G has a unit probe interval representation then it must not contain $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs.

Now we prove the other direction. Let $B_1, B_2, B_3, ...B_m$ be m ($m \geq 2$) groups of edge-consecutive 4-fans with all possible 2-leaves that are joined by straight 2-paths of length n ($n = 0, 1, 2....$) with all possible 2-leaves such that it has no $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. Let us again call this structure G. Thus G is a merge of I_1 without the forbidden subgraphs. We will now give a unit probe interval representation of G for different values of n.

Let B_i and B_j be any two groups of such 4-fans joined by such a straight 2-path of length n. Let us call this B_{ij} and it is isomorphic to I_1. From Figures 4.96, 4.89, 4.91, 4.90 it can be observed that the representation of the vertices on one end of a 4-fan or edge-consecutive 4-fans with all 2-leaves does not affect the vertex representation on the other end. So we can use the same strategy for representation on both ends and hence if the theorem holds for B_{ij} it should hold for the entire G. Hence it is sufficient to give a unit probe interval representation of B_{ij} for different values of n. Furthermore, since B_i and B_j have all possible 2-leaves and they are merge of edge consecutive 4-fans so the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf in its interior edge. Hence the last added vertex on the right end of B_i (if B_i is assumed to be constructed from the left) and the first added vertex on the left end of B_j must all be non-probes.

Case 1

$n = 0$: This is the case where the intermittent straight 2-path is of length 0. This implies that we have a merge of B_i and B_j such that the last edge of B_i coincides with the first edge of B_j. Both B_i and B_j are groups of edge consecutive 4-fans with every possible 2-leave. So we get a lengthier sequence of such edge-consecutive 4-fans whose length is equal to the sum of the lengths of B_i and B_j. By lemma 4.7.1 B_{ij} must have a unit probe interval representation.

Case 2

$n = 1$: This is the case where the intermittent straight 2-path is of length 1. This implies that we have a merge of B_i and B_j such that B_i and B_j shares exactly one vertex. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. So we get two sequences of edge-consecutive 4-fans sharing a common vertex whose length is equal to 1+length(B_i)+length(B_j). We have 2 cases here. The last added vertex of B_i is either the first added vertex of B_j or the last added vertex of B_i is adjacent to the first added vertex of B_j.

Let us first consider the case when the last vertex of B_i is the first vertex of B_j. If u_{n-1} and u_n are the last two added vertices of B_i in order and u_{n-2} is the center of the last 4-fan of B_i then by the representation in the Figure 4.96, $r(I_{u_{n-2}}) < r(I_{u_{n-1}}) < r(I_{u_n})$

or $r(I_{u_{n-2}}) < r(I_{u_n}) < r(I_{u_{n-1}})$ and u_n is a non-probe. If u_n and u_{n+1} are the vertices of the first K_2 in B_j and u_{n+2} is the third added vertex such that u_{n+1} is the center of the first 4-fan of B_j then in the representation $r(I_{u_n}) < r(I_{u_{n+1}}) < r(I_{u_{n+2}})$ or $r(I_{u_{n+1}}) < r(I_{u_n}) < r(I_{u_{n+2}})$ and u_n is a non-probe. Thus we can easily do unit probe representation of B_{ij} by doing B_i first and then continuing with the intervals B_j by making the last interval of B_i become the first interval of B_j such that $r(I_{u_{n-1}}) < r(I_{u_n}) < r(I_{u_{n+1}})$ as seen in the Figure 4.98.

Now let us consider the other case where the vertices are adjacent. In this case we get forbidden subgraphs $F2$ or $E2$ as shown in the Figure 4.99.

Case 3

$n = 2$: This is the case where the intermittent straight 2-path is of length 2. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. Length of B_{ij} is equal to 2+length(B_i)+length(B_j). We will look at the the last 4-fan of B_i and the first 4-fan of B_j with all possible 2-leaves in it. Since B_i and B_j have all possible 2-leaves so the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf in its interior edge. Hence the last added vertex on the right end of B_i (if B_i is assumed to be constructed from the left) and the first added vertex on the left end of B_j (B_j is also constructed from the left) must all be non-probes. The only possible structure that can form here is illustrated in Figure 4.100. In other cases we get adjacencies between non-probe vertices which gives us the forbidden subgraphs $F2, E8$ or $E10$. The structure in Figure 4.100 is possible because the right end 2-leaf from B_i (or u_n) is non-adjacent to the left end 2-leaf of B_j (or w_1) as shown in the same figure. We can do the unit probe representation of this structure as illustrated in Figure 4.100. Furthermore it can be easily seen that this structure is a just a staircase formed by two 4-fans. So other than the middle edges of the 4-fans ($u_n u_{n-1}$ and $w_1 w_2$, which forms the forbidden subgraph E1 with the 2-leaves) we can have 2-leaves on the other edge of the intermittent straight 2-path (which is $u_{n-1} w_2$ here).

Case 4

$n = 3$: This is the case where the intermittent straight 2-path is of length 3. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of such edge-consecutive 4-fans B_i and B_j joined by a straight 2-path of length 3 with all possible 2-leaves. Length of B_{ij} is equal to 3+length(B_i)+length(B_j). We will look at the last 4-fan of B_i and the first 4-fan of B_j with all possible 2-leaves in it. As discussed in the previous case, since B_i and B_j have all possible 2-leaves, then the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf on its interior edge. Hence the last added vertex on the right end of B_i and the first added vertex on the left end of B_j must all be non-probes. The only possible structures that can form here are illustrated in Figure 4.101. In other cases we get adjacencies between non-probe vertices which gives us the forbidden subgraph $E9$. The unit probe representations of the structures in the Figure 4.101 is given in the same figure. These structures are possible because the right end 2-leaf from B_i (or u_n) is non-adjacent to the left end 2-leaf of B_j (or w_1) as shown in the same figure. Furthermore it can be easily seen from these structures that we cannot have any 2-leaf on the interior edges of the intermittent 2-path in picture 1 of the same figure since that will form either an $E1$ or an $E2$, and we also can have exactly one 2-leaf on one interior edge of the intermittent 2-path (either on u_na or w_2a as shown in picture 2) otherwise we will get the forbidden subgraph $E6$ if both the 2-leaves were present, and we also will get either an $E7$ or $E2$ if there is a 2-leaf on u_nu_{n-1} or w_1w_2 respectively.

Case 5

$n = 4$: This is the case where the intermittent straight 2-path is of length 4. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of such edge-consecutive 4-fans joined by a straight 2-path of length 4 with all possible 2-leaves. The length of B_{ij} is equal to 4+length(B_i)+length(B_j).

As before, we will look at the the last 4-fan of B_i and the first 4-fan of B_j with all possible 2-leaves in it. Since B_i and B_j have all possible 2-leaves, then the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf in its interior edge. Hence the last added vertex on the right end of B_i (B_i, B_j are assumed to be constructed from the left and the intervals of vertices of B_i, B_j are placed from the left) and the first added vertex on the left end of B_j must all be non-probes. The only possible structures that can form here are illustrated in Figure 4.102. In other cases we either get adjacencies between non-probe vertices which gives us the forbidden subgraphs $E2, E3, E4, E5, E10$ or get an $F2$ in the underlying 2-path. The unit probe representations of the structures in Figure 4.102 is given in the same figure. The representation of these structures are possible because the right end 2-leaf from B_i is non-adjacent to the left end 2-leaf of B_j as shown in the same figure. Furthermore it can easily be seen from these structures that we cannot have any 2-leaf on the interior edges of the intermittent 2-path in picture 1 of the same figure since that will form either an $E3$ or an $E4$ and we also can have exactly one 2-leaf on one interior edge of the intermittent 2-path as shown in picture 2 otherwise we will get the forbidden subgraph $E2, E5, E6, E10$. The unit probe interval representation of 1 and 2 can be done by representing B_i first as in representations of figure 4.89, 4.90, 4.91 followed by the representation of the straight 2-path as in Figure 4.103 and finally representing B_j again in the same manner. Picture 3 is isomorphic to 2. The unit probe interval representation of picture 4 is given in the same figure. This is possible since the end 2-leaves of a 2-snake, extended-staircase and staircase can be made to have distinct end points which also can be seen from same figures 4.89, 4.90, 4.91. It can be noted here that Picture 4 is the only different case here whose representation is given in the same figure. This case is the structure containing a straight 2-path of length 10 (3 and 3 from the end 4-fans and 4 from the intermittent straight 2-path of length 4).

Furthermore on any part of the structure which has at least six consecutive K_3s, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3} otherwise we will get the forbidden subgraphs as shown in the picture 4 of the same Figure.

Case 6

$n > 4$: This is the case where the intermittent straight 2-path is of length greater than 4. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of such edge-consecutive 4-fans joined by a straight 2-path of length greater than 4 with all possible 2-leaves. It can be concluded from the previous case study that if we construct all the possible structures we can get two types of structures as given by the Figure 4.102.

Type1: These structures can be represented by doing B_i first as in the representations of the figure 4.89, 4.90, 4.91 followed by the representation of the straight 2-path as in the Figure 4.103 and finally representing B_j again in the same manner.

Type2: Structures joined by straight 2-paths of lengths at least 11 with all possible 2-leaves(at least 5 from the intermittent straight 2-paths and 3 each from the 4-fans on two ends). The 2-leaves on the interior edges of the straight 2-paths are present so that no forbidden subgraphs is formed. So as stated before, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3}. The unit probe interval representation of such structures are possible because the end 2-leaves of the 4-fans on two ends which are parts of the straight 2-path can be made to have distinct end points and then the intermittent straight 2-path can be represented as in the Figure 4.103.

Furthermore on any part of the structure which has at least six consecutive K_3s, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3} otherwise we will get forbidden subgraphs.

■

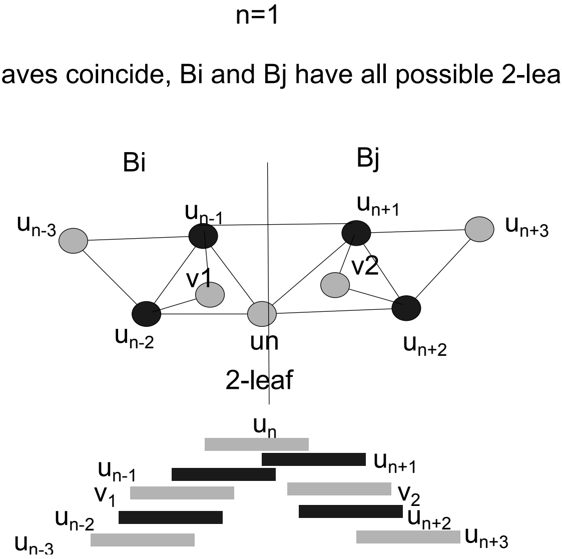


Figure 4.98: $n = 1 : I_1$

Now we look at interior 2-caterpillars isomorphic to I_2 which are derived from vertex consecutive 4-fans joined by straight 2-paths. In the following theorem we give a unit probe interval representation of such interior 2-caterpillars when they do not contain any of the forbidden subgraphs as induced subgraphs in it.

Theorem 4.7.5 *Interior 2-caterpillars given by structures isomorphic to merge of I_2 have unit probe interval representation if and only if they do not have E1, E2, E3, E4, E5, E6, E7, E8, E9, E9, E10, F2, F3, F4, F5, F6, F7, F3, F8, F9, F10, F11 as induced subgraphs.*

Proof: Let $B_1, B_2, B_3, ...B_m$ be m $(m \geq 2)$ groups of vertex-consecutive 4-fans with all possible 2-leaves that are joined by straight 2-paths of length n with all possible 2-leaves. This structure is isomorphic to merge of I_2 and we will call it G. If G has a unit probe interval representation then it must not contain $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs by previous results.

Now for the other direction let$B_1, B_2, B_3, ...B_m$ be m $(m \geq 2)$ groups of vertex-

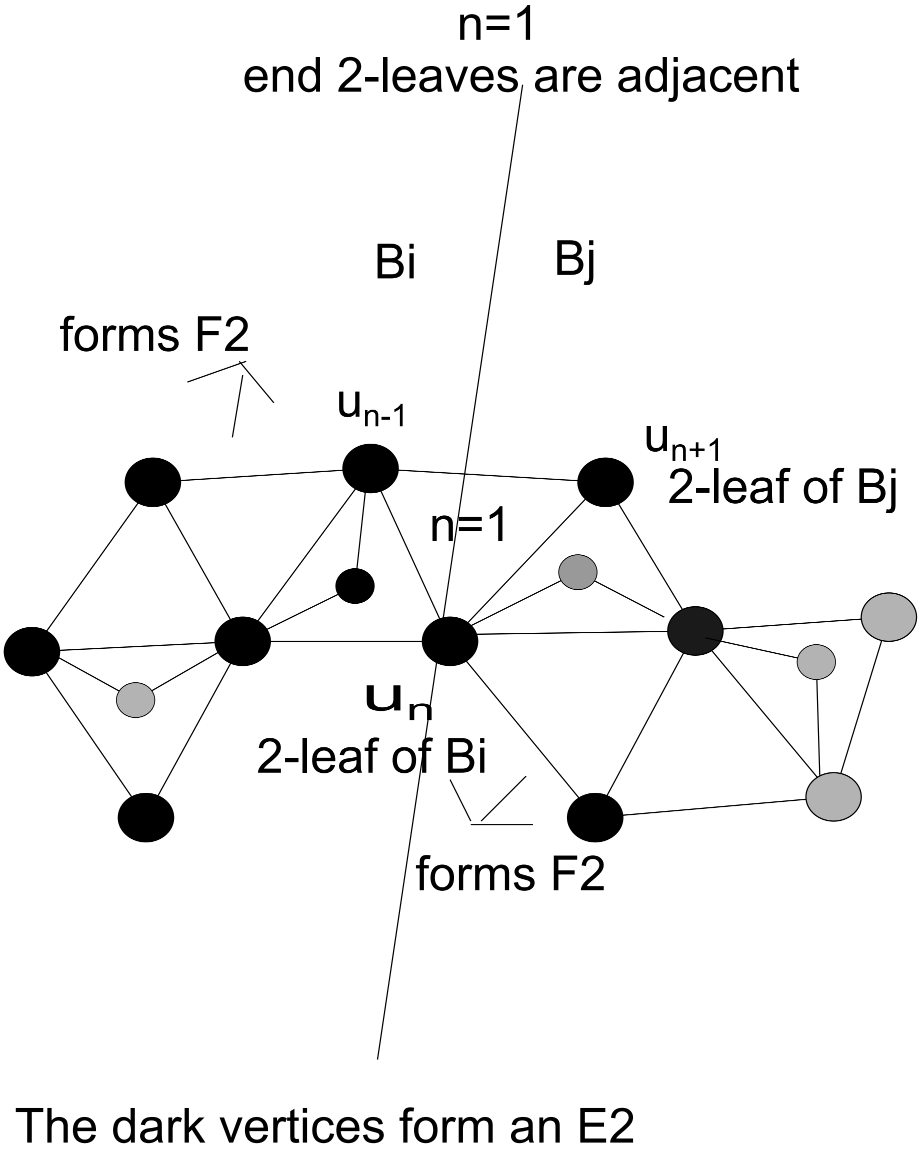


Figure 4.99: $n = 1 : I_1$

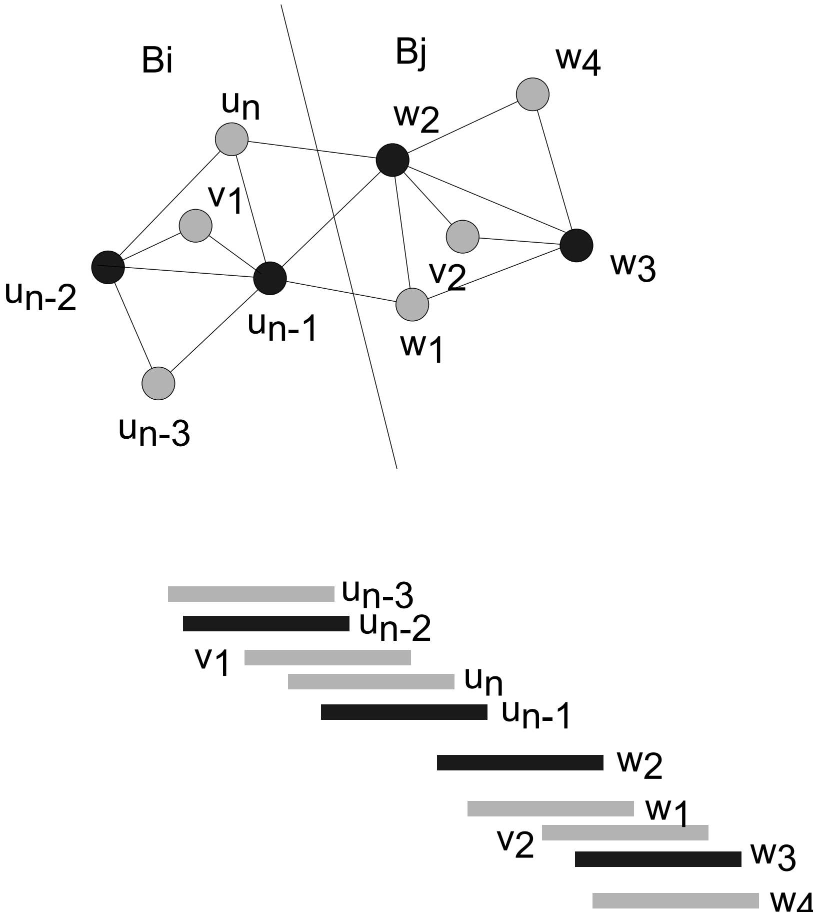


Figure 4.100: $n = 2 : I_1$

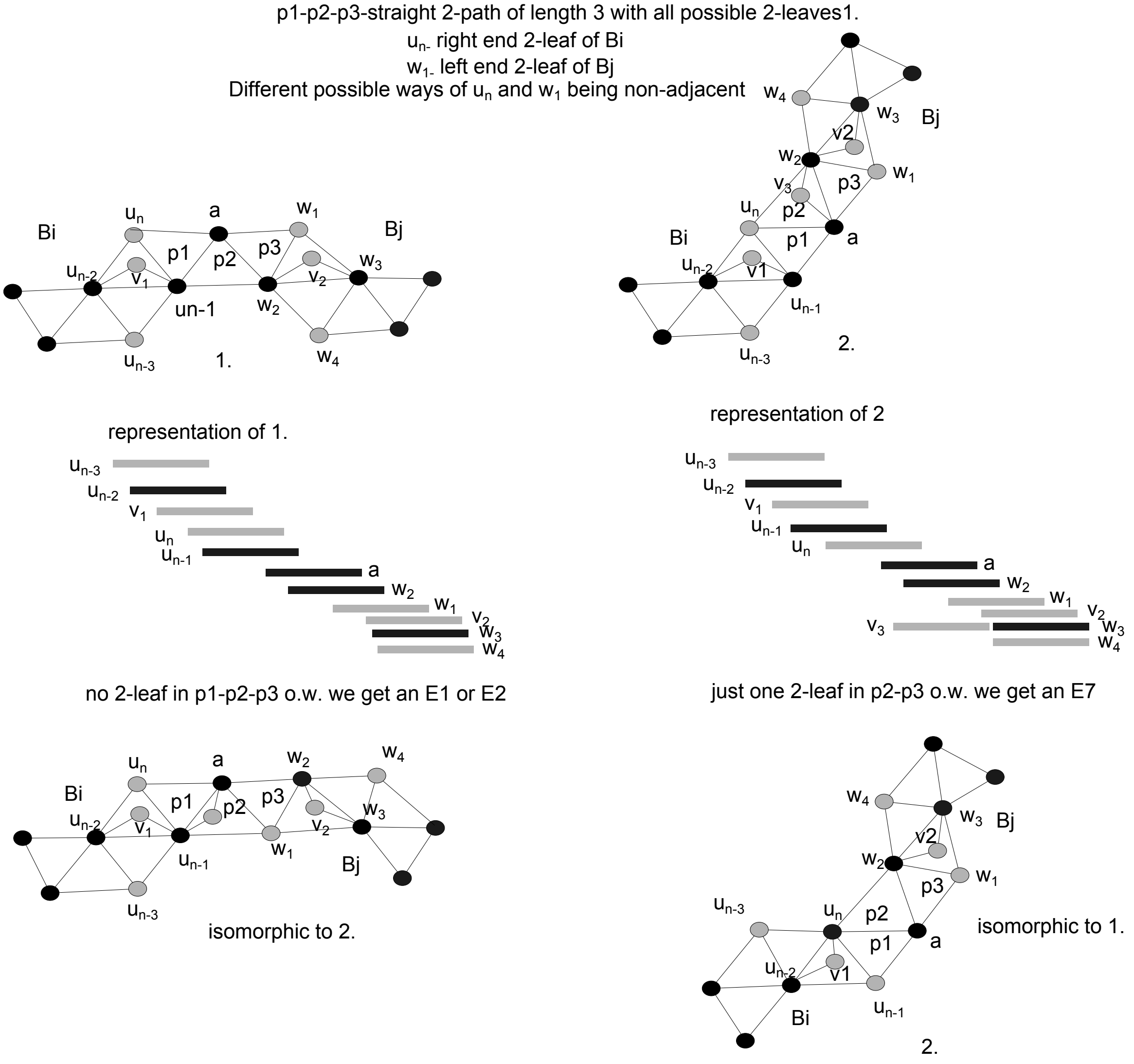
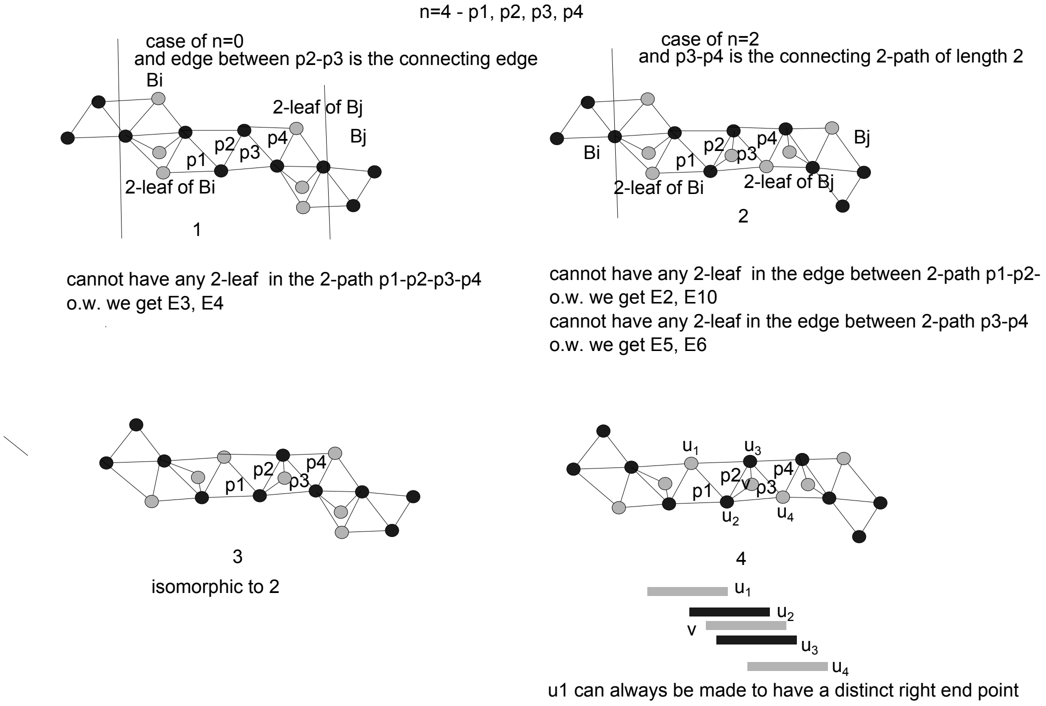


Figure 4.101: $n = 3 : I_1$



n=4 - p1, p2, p3, p4

case of n=0
and edge between p2-p3 is the connecting edge

case of n=2
and p3-p4 is the connecting 2-path of length 2

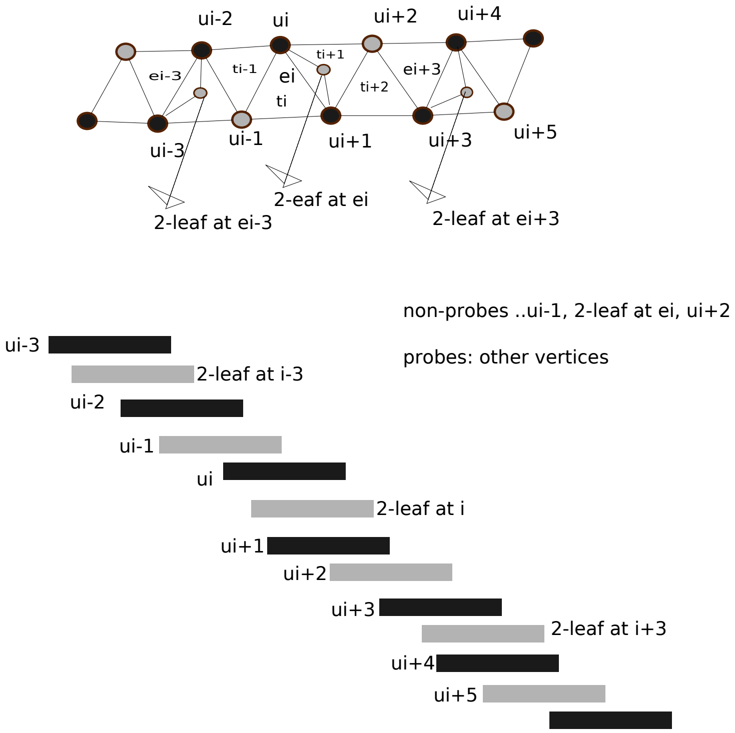
cannot have any 2-leaf in the 2-path p1-p2-p3-p4
o.w. we get E3, E4

cannot have any 2-leaf in the edge between 2-path p1-p2-
o.w. we get E2, E10
cannot have any 2-leaf in the edge between 2-path p3-p4
o.w. we get E5, E6

isomorphic to 2

u1 can always be made to have a distinct right end point

Figure 4.102: $n = 4 : I_1$



non-probes ..ui-1, 2-leaf at ei, ui+2

probes: other vertices

Figure 4.103: Representation of straight 2-path with all 2-leaves

consecutive 4-fans with all possible 2-leaves that are joined by straight 2-paths of length n again with all possible 2-leaves such that there are no $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$ or $F11$ as induced subgraphs. Let us again call this structure G. We will now give a unit probe interval representation of G for different values of n.

Let B_i and B_j be any two groups of such 4-fans joined by a such a straight 2-path of length n. Let us call this B_{ij}, and it is isomorphic to I_2. It is easy to see that if the theorem holds for B_{ij} it should hold for the entire G since representation of the vertices of one end of any B_i is not dependent on the representation of vertices of the other end of B_i for any i as we have seen earlier. So we give a unit probe interval representation of B_{ij} for different values of n. Furthermore, since B_i and B_j have all possible 2-leaves and they are merge of vertex consecutive 4-fans so the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf on its interior edge. Hence, as in the previous theorem, the last added vertex on the right end of B_i and the first added vertex on the left end of B_j must all be non-probes.

Case 1

$n = 0$: This is the case where the intermittent straight 2-path is of length 0. This implies that we have B_i and B_j such that the last edge of B_i coincides with the first edge of B_j. Both B_i and B_j are groups of edge consecutive 4-fans with all possible 2-leaves. So we get two possible cases here. Either the right end 2-leaf of B_i is adjacent to the left end 2-leaf of B_j or the 2-leaves coincide. If the 2-leaves are adjacent then we get the forbidden subgraph $E3$ as shown in the Figure 4.104. If they coincide then we either get an $F2$ or a unit probe representation as shown in the same figure.

Case 2

$n = 1$: This is the case where the intermittent straight 2-path is of length 1. This implies that we have B_i and B_j such that B_i and B_j shares exactly one vertex. Both B_i and B_j are groups of vertex consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of vertex-consecutive 4-fans sharing a common vertex whose length is equal to 1+length(B_i)+length(B_j). We have 2 cases here. The last added vertex of B_i (B_i, B_j is assumed to be constructed from the left without loss of generality) is either the first added vertex of B_j or they are adjacent.

Let us first consider the case when the right end vertex of B_i is the left end vertex of B_j. This is just another case of vertex consecutive 4-fans since B_i is vertex consecutive to B_j as seen in Figure 4.105 and so we have a unit probe representation of this.

Now let us consider the other case where the vertices are adjacent. In this case we get the forbidden subgraph $F2$ in the underlying 2-paths or an $E2$ as shown in Figure 4.105.

Case 3

$n = 2$: This is the case where the intermittent straight 2-path is of length 2. This implies that we have B_i and B_j such that B_i and B_j shares nothing. Both B_i and B_j are groups of vertex-consecutive 4-fans with all possible 2-leaves. The length of B_{ij} is equal to 2+length(B_i)+length(B_j). We will look at the the last 4-fan of B_i and the first 4-fan of B_j with all possible 2-leaves on it. Since B_i and B_j have all possible 2-leaves, then the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf on its interior edge. Hence the last added vertex on the right end of B_i (B_i, B_j is assumed to be constructed from the left) and the first added vertex on the left end of B_j must all be non-probes. The structures that can be formed here are illustrated in the Figure 4.106. If the right end 2-leaf of B_i is adjacent to the left end 2-leaf of B_j then we get forbidden subgraphs $E10$ or $E8$ as shown in the same Figure. In the other case when the right end 2-leaf from B_i (or u_n) is non-adjacent to the left end 2-leaf of B_j (or w_1)we get a structure as shown in Figure 4.106. We do the unit probe representation of this structure as illustrated in the same Figure.

Case 4

$n = 3$: This is the case where the intermittent straight 2-path is of length 3. This

implies that we have B_i and B_j such that B_i and B_j shares nothing. Both B_i and B_j are groups of vertex-consecutive 4-fans with all possible 2-leaves. The length of B_{ij} is equal to 3+length(B_i)+length(B_j). We will look at the the last 4-fan of B_i and the first 4-fan of B_j with all possible 2-leaves. Since B_i and B_j have all possible 2-leaves so the last and first K_3s of the underlying 2-path of both B_i and B_j must have a 2-leaf on its interior edge. Hence the last added vertex on the right end of B_i and the first added vertex on the left end of B_j must all be non-probes like the previous cases. The only possible structures that can have a unit probe interval representation here are illustrated in Pictures 1 and 2 of Figure 4.101. Furthermore picture 1 cannot even have any 2-leaf on the interior edges of the intermittent straight 2-path otherwise forbidden subgraphs $E1$ or $E2$ are formed whereas picture 2 can have just one 2-leaf in the straight 2-path as shown in the same Figure. Otherwise forbidden subgraph $E7$ is formed if more than one 2-leaf is attached to the straight 2-path. In the other case we get adjacencies between non-probe vertices which gives us the forbidden subgraph $E9$ or $F2$ in the underlying 2-path.

Case 5

$n = 4$: This is the case where the intermittent straight 2-path is of length equal to 4. Both B_i and B_j are groups of vertex-consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of such vertex-consecutive 4-fans joined by a straight 2-path of length equal to 4 with all possible 2-leaves. The 2-leaves on the interior edges of the straight 2-paths are present so that no forbidden subgraph is formed. So as stated before, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3}. Let u_{n-1}, u_n be the last two added vertices of B_i such that u_n is the right end 2-leaf of B_i. We know that u_n must be non-probe and u_{n-1} must be a probe. Furthermore either u_n or u_{n-1} can be made to have a distinct right end point as required. The same can be concluded about the left end 2-leaf of B_j. Let w_1, w_2 be the first two added vertices of B_j such that w_1 is the left end 2-leaf of B_j. We know that w_1 must be non-probe and w_2 must be a probe. Furthermore either w_1 or w_2 can be made to have a distinct left end point as required. Also, since the intermittent straight 2-path is of length atleast 4, there exists at least one vertex from the 2-path between u_{n-1}, u_n and w_1, w_2 such that the end 2-leaves from B_i and B_j are never adjacent. We look at how the last 4-fan from B_i connects to the first 4-fan from B_j via the straight 2-path of length 4. The different possible structures are shown in Figure 4.102. Furthermore on any part of the structure which has at least six consecutive K_3s, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3} otherwise we will get the forbidden subgraphs. The unit probe interval representations of 1 and 2 are done by doing B_i first as in figure 4.97 followed by the representation of the straight 2-path as in Figure 4.103 and finally representing B_j again as in figure 4.97. The graph in Picture 3 is isomorphic tothe graph in picture 2. The unit probe interval representation of the graph in picture 4 is given in the same figure. In all 3 representations we use one of the 2 representations from Figure 4.97 for B_i and B_j.

Case 6

n >4: This is the case where the intermittent straight 2-path is of length greater than 4. This implies that we have B_i and B_j such that B_i and B_j shares no vertex. Both B_i and B_j are groups of vertex-consecutive 4-fans with all possible 2-leaves. So we get 2 sequences of such vertex-consecutive 4-fans joined by a straight 2-path of length greater than 4 with all possible 2-leaves. It can be concluded from the previous case study that if we construct all the possible structures we will never get adjacencies between the end 2-leaves of B_i and B_j and they would have at least 2 intermittent vertices between them. The 2-leaves on the interior edges of the straight 2-paths are present so that no forbidden subgraph is formed. So as stated before, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3}. Let u_{n-1}, u_n be the last two added vertices of B_i such that u_n is the right end 2-leaf of B_i. We know that u_n must be a non-probe and u_{n-1} must be a probe. Furthermore either u_n or u_{n-1} can

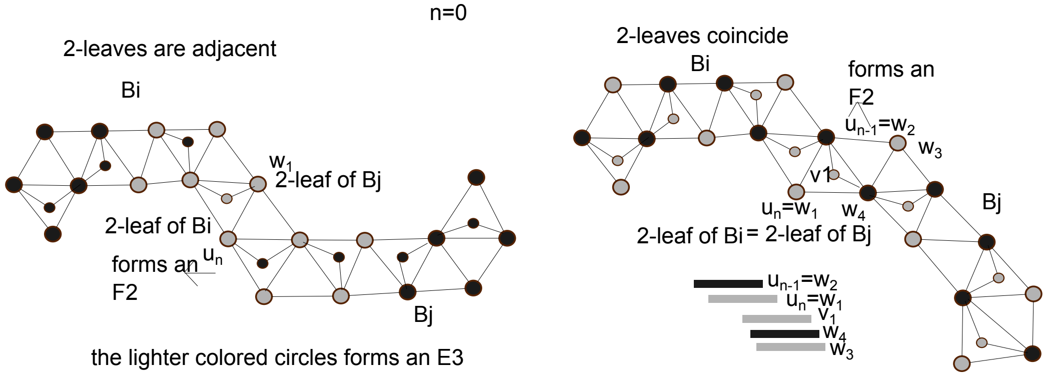


Figure 4.104: $n = 0 : I_2$

be made to have a distinct right end point as required. The same can be concluded about the left end 2-leaf of B_j. Let w_1, w_2 be the first two added vertices of B_j such that w_1 is the left end 2-leaf of B_j. We know that w_1 must be non-probe and w_2 must be a probe. Furthermore either w_1 or w_2 can be made to have a distinct left end point as required. Also, since the intermittent straight 2-path is of length atleast 4 there exists at least two vertices from the 2-path between u_{n-1}, u_n and w_1, w_2 such that the end 2-leaves from B_i and B_j are never adjacent. Like the previous case, the unit probe interval representation of such structures are done by representing B_i first (as in figure 4.97) then the intermittent straight 2-path can be represented as in Figure 4.103 followed by B_j (as in figure 4.97).

■

Now we look at interior 2-caterpillars isomorphic to I_3 which are derived from vertex consecutive 4-fans joined by straight 2-paths with edge-consecutive 4-fans. In the following theorem we give unit probe interval representations of such interior 2-caterpillars when they do not contain certain forbidden subgraphs as induced subgraphs.

Theorem 4.7.6 *Interior* 2*-caterpillars given by structures isomorphic to merge of* I_3 *have unit probe interval representation if and only if they do not have E1, E2, E3,*

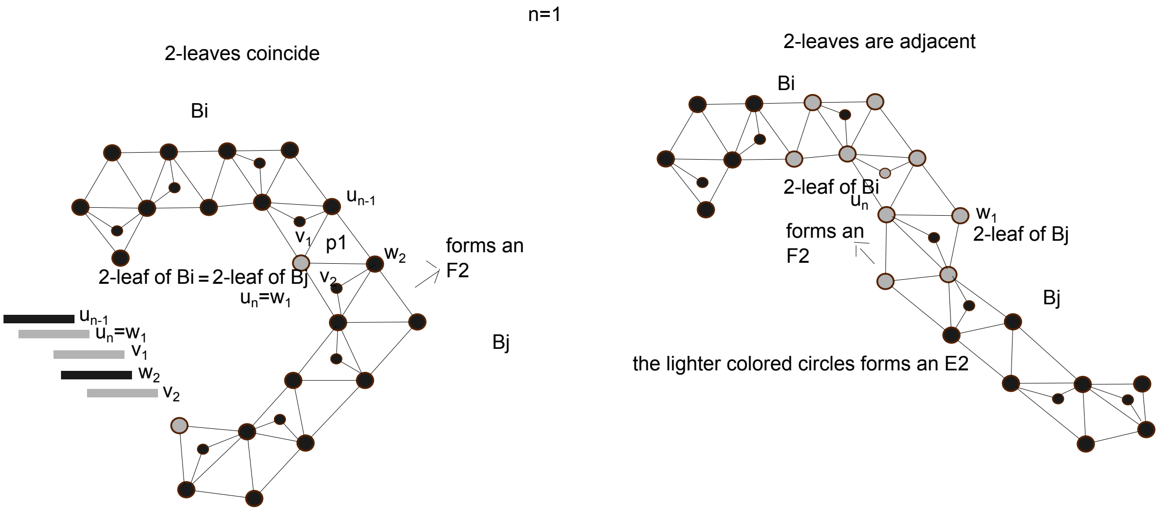


Figure 4.105: $n = 1 : I_2$

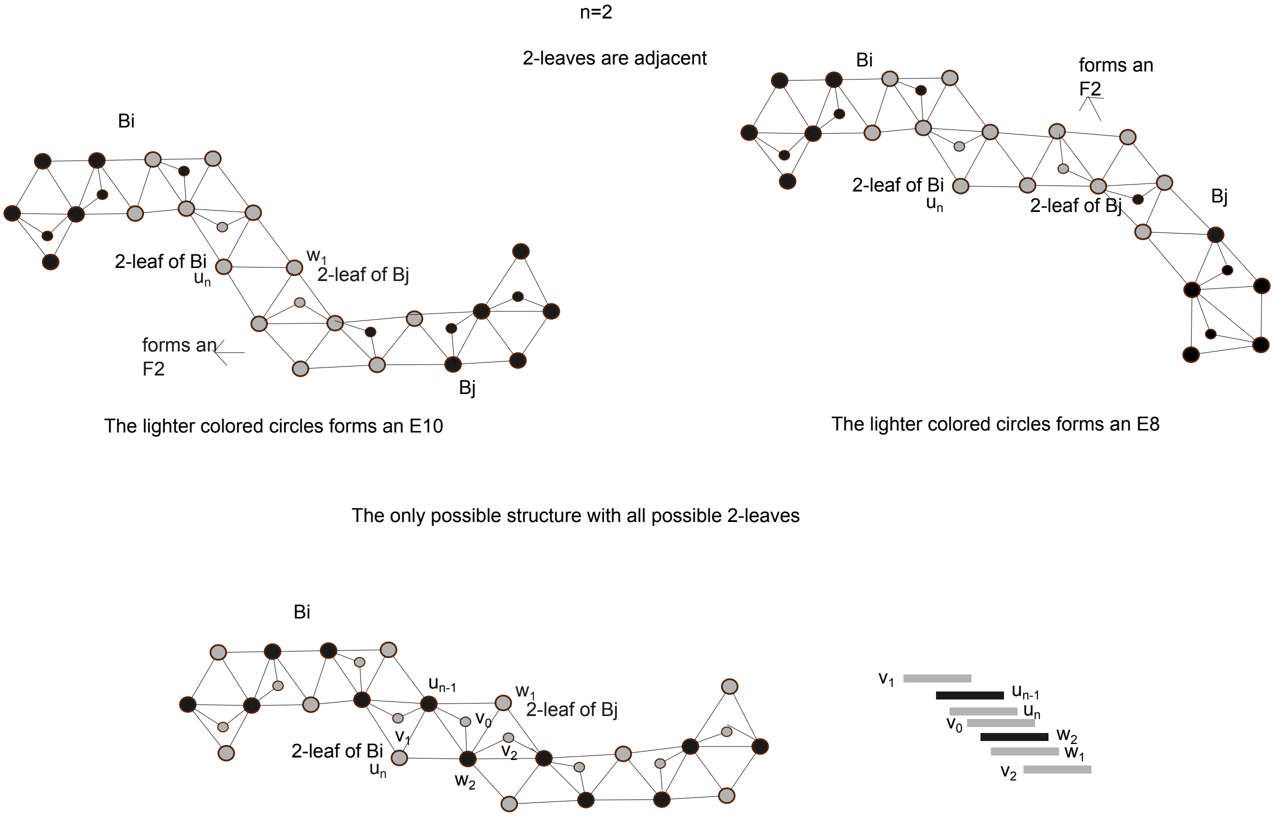


Figure 4.106: $n = 2 : I_2$

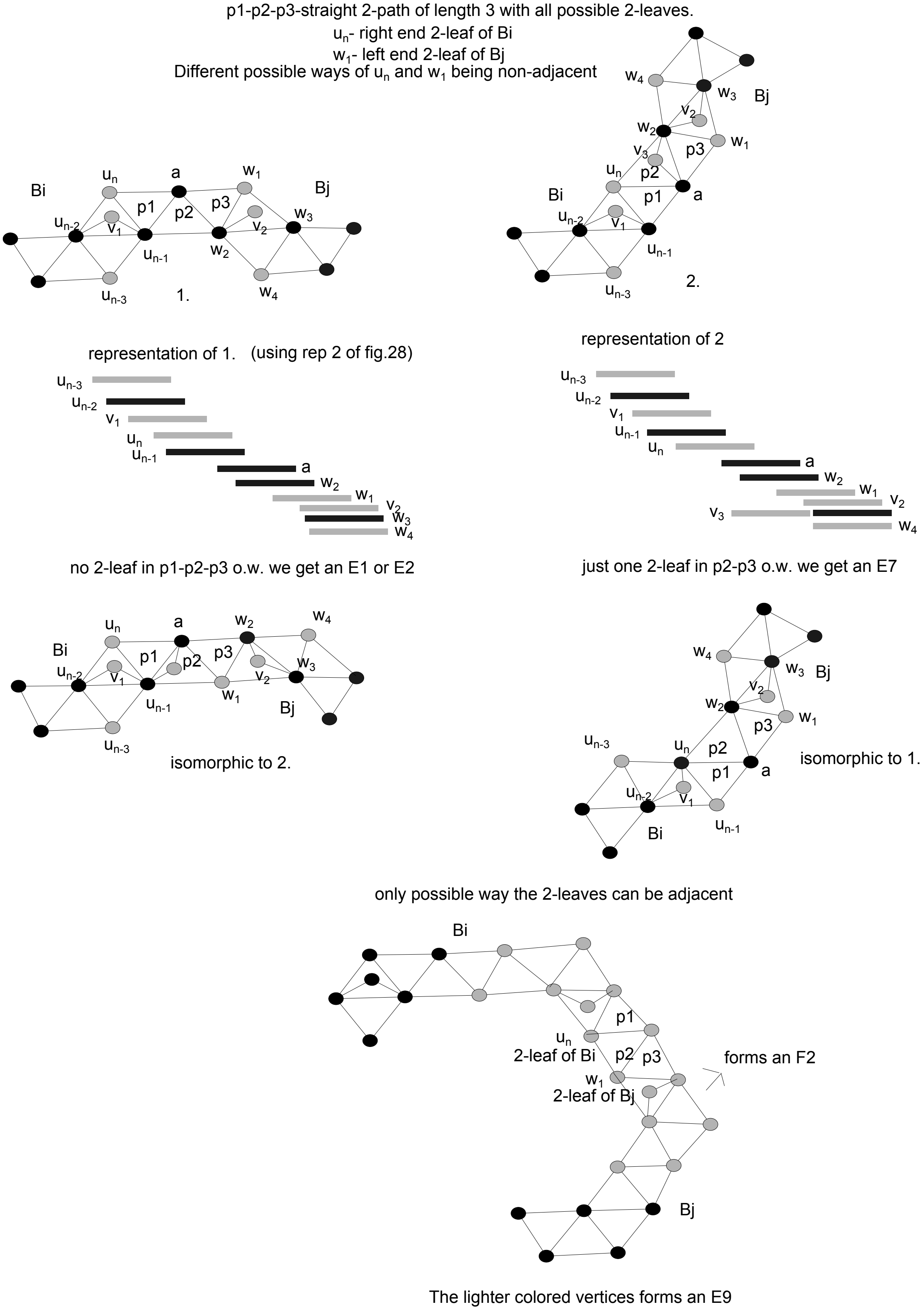

p1-p2-p3-straight 2-path of length 3 with all possible 2-leaves.
u_n- right end 2-leaf of Bi
w_1- left end 2-leaf of Bj
Different possible ways of u_n and w_1 being non-adjacent
Bi
Bj
u_n
a
w_1
p1
p2
p3
u_n-2
v_1
v_2
w_3
u_n-1
w_2
u_n-3
w_4
1.
2.
representation of 1. (using rep 2 of fig.28)
representation of 2
no 2-leaf in p1-p2-p3 o.w. we get an E1 or E2
just one 2-leaf in p2-p3 o.w. we get an E7
isomorphic to 2.
isomorphic to 1.
only possible way the 2-leaves can be adjacent
u_n 2-leaf of Bi
w_1 2-leaf of Bj
forms an F2
The lighter colored vertices forms an E9

Figure 4.107: $n = 3 : I_2$

E4, E5, E6, E7, E8, E9, E9, E10, F2, F3, F4, F5, F6, F7, F3, F8, F9, F10, F11 as induced subgraphs.

Proof:

Let G_1 and G_2 be groups of vertex-consecutive and edge-consecutive 4-fans respectively joined by a straight 2-path of length n, with all possible 2-leaves. Let us call this graph G. This structure is isomorphic to I_3. If G has a unit probe interval representation then it must not contain $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs.

Now for the other direction let G_1 and G_2 be groups of vertex-consecutive and edge-consecutive 4-fans respectively joined by a straight 2-path of length n, with all possible 2-leaves. Let us call this graph G. Hence G is isomorphic to I_3. G does not contain $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. We will give a unit probe interval representation of G. We know that any number of edge-consecutive 4-fans with all possible 2-leaves (that form a 2-path without the 2-leaves) have unit probe interval representation if they do not have $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E8$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. We also know that any number of groups of vertex-consecutive 4-fans with all possible 2-leaves (that forms a 2-path without the 2-leaves) have unit probe interval representation if they do not have F2, E1, E5, E6, E7 as induced subgraphs. Let v_{n-1} and v_n be the last 2 added vertices of $G1$ and u_1, u_2 be the first K_2 of $G2$. Thus v_n is the right end 2-leaf of the underlying 2-path of $G1$ and u_1 is the left end 2-leaf of the underlying path of $G2$

Case 1($n = 0$): The last 2 added vertices of G_1 must be the first K_2 of $G2$.

Since we consider G_1 and G_2 with all possible 2-leaves in them, the last and first K_3 of $G1$ and $G2$ respectively must have 2-leaves on its interior edges. Hence both v_n and u_1 must be non-probes. Hence if v_n is adjacent to u_1 then we get a problem in the representation of G as we get the forbidden subgraph $E3$ as an induced subgraph in it as shown in Figure 4.108. Otherwise $v_n = u_1$ and we get a unit probe representation as shown in Figure 4.108

Case $2(n = 1)$: If there exists exactly one K_3 between G_1 and G_2 then we have to consider two subcases. Either the 2-leaves coincide which is ok as shown in the figure 4.98 or they are adjacent which forms an induced $E2$ as shown in Figure 4.109.

Case $3(n = 2)$: If there exists two K_3 between G_1 and G_2 then we get the following subcases. The possible representable structures and their representations are given in the Figure 4.100. The non-possiblity occurs when the non-probe 2-leaves of $G1$ and $G2$ are adjacent and we get forbidden subgraphs $F2, E8, E10$ as shown in Figure 4.110.

Case $4(n = 3)$: If there exists three K_3 between G_1 and G_2 then we get the following subcases. The possible representable structures and their representations are given in Figure 4.101 (Since G_1 is vertex-consecutive and G_2 is edge-consecutive, we use representation 2 of Figure 4.97 for G_1 for picture 1 and representation 1 of Figure 4.97 for G_1 for the picture 2. Furthermore we use representations for Figures 4.89, 4.90, 4.91 for G_2 in both pictures 1 and 2 depending on the kind of G_2 that is present there. Similar to the previous cases the non-possibility occurs when the non-probe 2-leaves of G_1 and G_2 are adjacent and we get forbidden subgraphs $F2$ or $E9$.

Case $5(n = 4)$: If there exists 4 K_3 between G_1 and G_2 then we get the following subcases. Since G_1 and G_2 have all possible 2-leaves so the last and first K_3s of the underlying 2-path of both G_1 and G_2 must have a 2-leaf in its interior edge. Hence the last added vertex on the right end of G_1 (if G_1 is assumed to be constructed from the left) and the first added vertex on the left end of G_2 (if G_2 is assumed to be constructed from the left) must all be non-probes. The only possible structures that can form here are illustrated in the Figure 4.102. In other cases we either get adjacencies between non-probe vertices which gives us the forbidden subgraph $E2, E3, E4, E5, E10$ or get an $F2$ in the underlying 2-path. The unit probe representations of the structures in

Figure 4.102 are given in the same figure. These structures are possible to represent because the right end 2-leaf from G_1 is non-adjacent to the left end 2-leaf of G_2 as shown in the same figure. Furthermore it can be easily seen from these structures that we cannot have any 2-leaf on the interior edges of the intermittent 2-path in picture 1 of the same figure since that will form either an $E3$ or an $E4$ and we also can have exactly one 2-leaf on one interior edge of the intermittent 2-path as shown in picture 2 otherwise we will get the forbidden subgraph $E2, E5, E6, E10$. Pictures 1 and 2 are represented by doing the representation of G_1 first as in the Figure 4.97 followed by the straight 2-path as in the Figure 4.103 and then finally doing the representation of G_2 as in the Figures 4.89, 4.90, 4.91. Picture 4 is the only different case here whose representation is given in the same figure. This structure contains a straight 2-path of length 10 (3 and 3 from the end 4-fans and 4 from the intermittent straight 2-path of length 4). Here too we do the representation of G_1 first as in picture 1 of the Figure 4.97 followed by the straight 2-path as in the Figure 4.103 and then finally doing the representation of G_2 as in the Figures 4.89, 4.90, 4.91 so that the left end 2-leaves of G_2 have distinct end points (in all of these representations we can make the end 2-leaves of the underlying 2-path have distinct end points). Furthermore on any part of the structure which has at least six consecutive K_3s, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3} otherwise we will get the forbidden subgraphs. So avoiding such 2-leaves we can easily get the unit probe interval representation of this last subcase.

Case 6(n >4): If there exists more than four K_3's between G_1 and G_2 then as before we get three subcases. This is the case where the intermittent straight 2-path is of length greater than 4. It can be concluded from the previous case study that if we construct all the possible structures as we did for case 5 we can either get any structure whose representation can be done by representing G_1 first as in Figure 4.97 followed by the straight 2-path as in Figure 4.103 and then finally doing the represen-

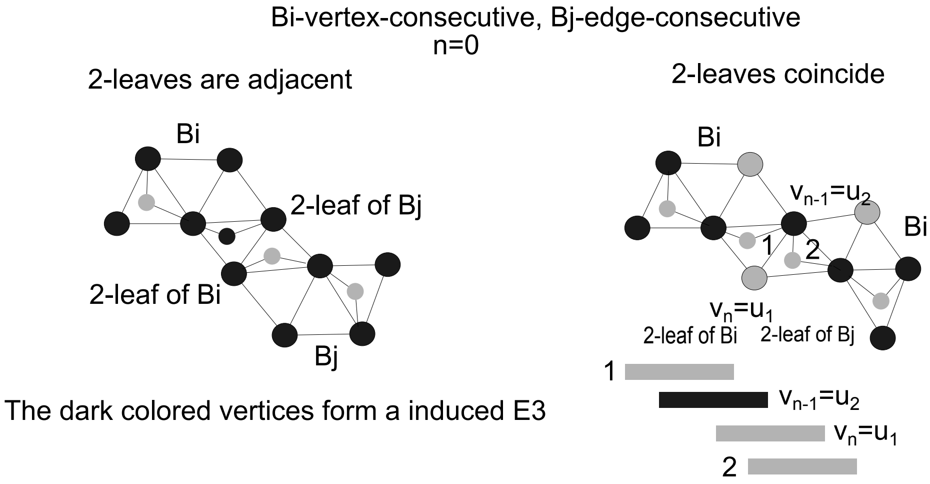


Figure 4.108: $n = 0 : I_3$

tation of G_2 as in Figures 4.89, 4.90, 4.91 or structures joined by straight 2-paths of lengths at least 11 (at least 5 from the intermittent straight 2-paths and 3 each from the 4-fans on two ends). The 2-leaves on the interior edges of the straight 2-paths are present so that no forbidden subgraphs is formed. So as stated before, if edge e_k has a 2-leaf then its nearest 2-leaf must be on the edge e_{k+3}, e_{k-3}. The unit probe interval representation of such structures are possible because the end 2-leaves of the 4-fans on two ends which are parts of the straight 2-path can be made to have distinct end points (by the second pictures of Figures 4.89, 4.90 and easy to see in 4.91 and by representation in Figure 4.97) and the intermittent straight 2-path can be represented as in Figure 4.103.

By the same argument used in the proofs of finding an unit probe interval representation of I_1 and I_2, it can be concluded that the results also holds for merge of I_3.

■

Note: This note is valid for all the above 3 theorems. Let us consider some B_{ij} where B_i and B_j are joined by some straight 2-path of length p. Suppose that this B_{ij} fails to have a representation at the merge due to adjacency of non-probes. It is

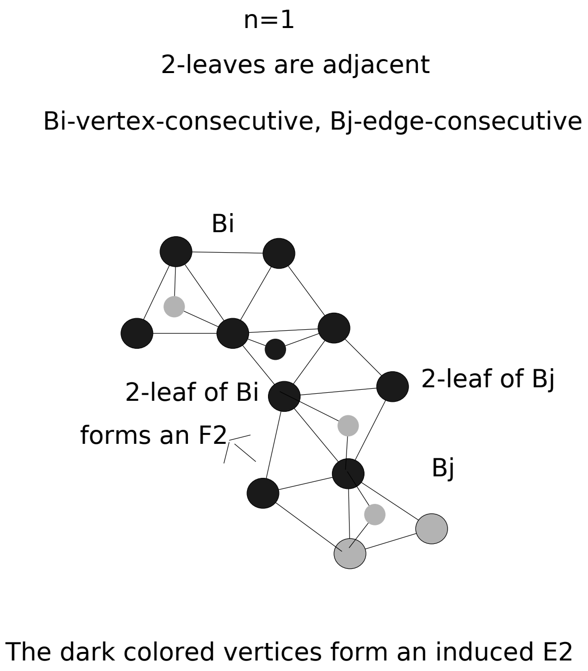

n=1
2-leaves are adjacent
Bi-vertex-consecutive, Bj-edge-consecutive
Bi
2-leaf of Bi
2-leaf of Bj
forms an F2
Bj
The dark colored vertices form an induced E2

Figure 4.109: $n = 1 : I_3$

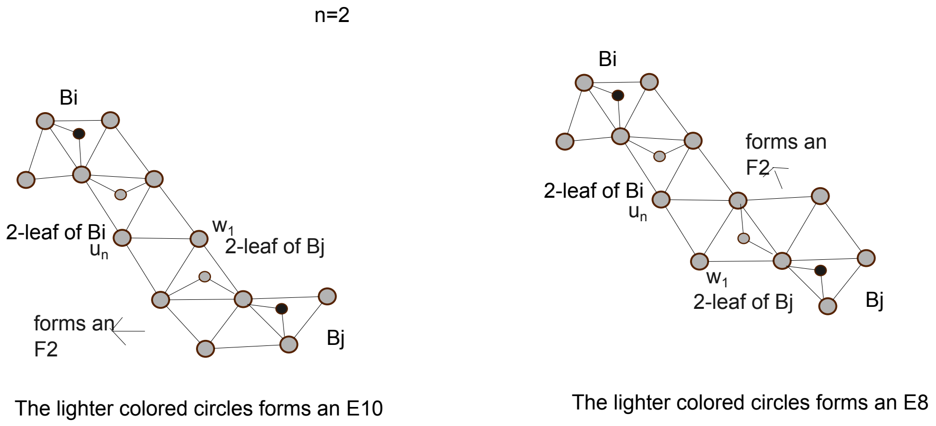
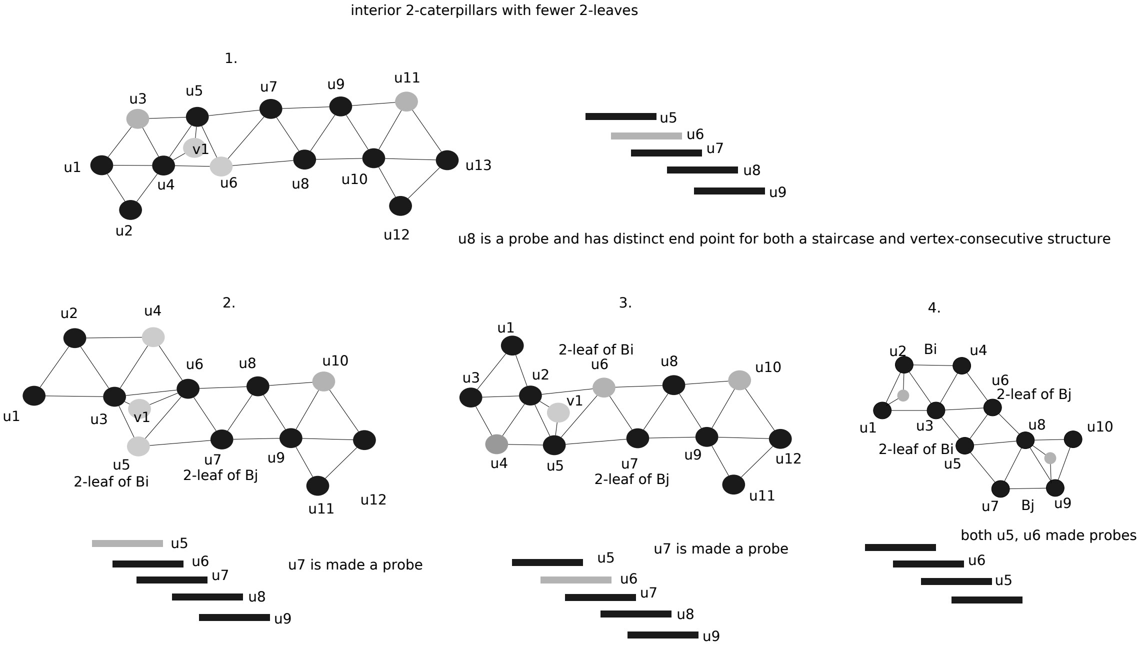


Figure 4.110: $n = 2 : I_3$



u8 is a probe and has distinct end point for both a staircase and vertex-consecutive structure

u7 is made a probe

u7 is made a probe

both u5, u6 made probes

Figure 4.111: Interior 2-caterpillars with fewer 2-leaves

easy to see that this same graph will be unit probe interval representable if, at the merge, one 2-leaf is taken off from either B_i or B_j (sometimes from both B_i and B_j as in picture 4 of the Figure 4.111), and the last two 4-fans of that B_i or B_j (or both B_i and B_j) are isomorphic to either a staircase or a vertex-consecutive structure so that the non-probe restriction from its end 2-leaf is removed and their intervals can be made to have distinct end points. In the other cases we get the forbidden subgraphs as seen in Figure 4.87.

These structures can be looked as merge of interior 2-caterpillars with fewer 2-leaves or I_i, $(i = 1, 2, 3)$ with fewer 2-leaves. Their representations are given in the Figure 4.111. It is worth mentioning here that we can also get other kinds of interior 2-caterpillars like interior 2-caterpillars which are just subgraphs of I_1, I_2, I_3. All these structures will also have unit probe interval representation since from all the previous representations we know that all the 2-leaves are non-probes and their removal from the representation will not affect the interval layout of the underlying 2-path.

We now have a characterization for interior 2-caterpillars isomorphic to I_1, I_2, I_3 which are unit probe interval graphs. Since an interior 2-caterpillar is either isomorphic to one of the above I_i where $(i = 1, 2, 3)$, their subgraphs or I_i's with fewer 2-leaves, we can now easily characterize interior 2-caterpillars which are unit probe interval graphs. The following theorem gives a characterization of interior 2-caterpillars which are unit probe interval graphs.

Theorem 4.7.7 *An interior* 2*-caterpillar is a unit probe interval graph if and only if it does not contain E1, E2, E3, E4, E5, E6, E7, E8, E9, E9, E10, Group-E11, F2, F3, F4, F5, F6, F7, F8, F9, F10, F11 as generated subgraphs.*

Proof: Let G be an interior 2-caterpillar which is a unit probe interval graph. Since $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ do not have unit probe interval representation, G cannot have them as induced subgraphs.

For the other direction, let G be any interior 2-caterpillar without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$as generated subgraphs. Any interior 2-caterpillar can be considered as groups of edge or vertex consecutive 4-fans joined together by straight 2-paths with all possible 2-leaves or fewer 2-leaves. So G can be assumed to be groups of edge or vertex consecutive 4-fans joined together by straight 2-paths with all possible or fewer 2-leaves such that nowhere $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ are formed as induced subgraphs. Thus G is either I_1, I_2, I_3, their subgraphs or I_i's (i=1,2,3) with fewer 2-leaves without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$. Therefore by the previous three theorems and the above note it must have a unit probe interval representation. ■

The next theorem derives a relationship between 2-tree unit probe interval graphs and interior 2-caterpillars.

Theorem 4.7.8 *A 2-tree is a unit probe interval graph if and only if it is an interior 2-caterpillar without E1, E2, E3, E4, E5, E6, E7, E8, E9, E9, E10, Group-E11, F1, F2, F3, F4, F5, F6, F7, F8, F9, F10, F11 as generated subgraphs.*

Proof: This proof is an extension of previous results. We first prove the necessary part first. Let G be 2-tree UPIG. By Theorem 4.3.3 we have that G must be an interior 2-caterpillar. Since $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ are not UPIG, G cannot contain any of them as induced subgraphs. So G must be an interior 2-caterpillar without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$.

For the other direction let us now assume that G is a 2-tree interior 2-caterpillar without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs. By the previous theorem, an interior 2-caterpillar is a unit probe interval graph if and only if it does not contain $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as generated subgraphs. Since G is an interior 2-caterpillar without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ it must be a unit probe interval graph.

■

Finally the next theorem gives a complete characterization of 2-trees which are unit probe interval graphs. The total number of forbidden subgraphs is 27.

Theorem 4.7.9 *A 2-tree is a unit probe interval graph if and only if it does not contain E1, E2, E3, E4, E5, E6, E7, E8, E9, E9, E10, Group-E11, F1, F2, F3, F4, F5, F6, F7, F8, F9, F10, F11 as generated subgraphs.*

Proof: Let G be a 2-tree. let us first assume that it is a unit probe interval graph. Since $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ do not have unit probe interval interval representation, G cannot contain them as induced subgraphs.

For the other direction let us now assume that G is a 2-tree without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as induced subgraphs and that G is not a unit probe interval graph. Then by the previous theorem G is not an interior 2-caterpillar without $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$ as generated subgraphs. But it is given that G does not contain any induced $E1$, $E2$, $E3$, $E4$, $E5$, $E6$, $E7$, $E8$, $E9$, $E10$, Group-$E11$, $F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$, $F8$, $F9$, $F10$, $F11$. So G must not be an interior 2-caterpillar. So it either is a non-interior 2-caterpillar or not a 2-caterpillar at all. Thus G either has an $F1$ or G has $B1, B1', B2, B3$ as induced subgraphs. But G cannot have an $F1$ as an induced subgraph so it must have $B1, B1', B2, B3$ as induced subgraphs. It is easy to see from Figure 4.9 that $B1, B1'$ have $E1$ and $B2, B3$ have $F1$ in it respectively as induced subgraphs. This implies that G has $F1$ or $E1$ as induced subgraphs which is a contradiction. Hence G must be a unit probe interval graph. ■

Conclusion:-

The preceding results compliment the existing characterizations of subclasses of probe interval graphs which include cycle-free [41], unit cycle-free [15] and bipartite unit [10]. These results also add to the spirit of the results of Corneil and Pržulj [34] in showing that the complete characterization of even restricted subclasses of probe interval graphs will be a challenging problem.

Some progress, however, has been made in fnding classes of probe interval graphs which will admit to an (albeit complicate) characterization via forbidden induced subgraphs. See for example the recent work of Brown, Busch, and Isaak [5], and of

Brown, Flesch, and Lundgren [7].

REFERENCES

[1] S. Benzer, "On the topology of the genetic fine structure", *Proceedings of the National Academy of Sciences of the United States of America* **45 (11)** 1959, 1607-1620

[2] L. W. Beineke and R. E. Pippert, Properties and characterization of k-trees, *Mathematika* **18** (1971), 141-151.

[3] J. J. Beyerl, R. E. Jamison, Interval Graphs with Containment Restrictions *Congressus Numberantium* **191** (2008), 117-128.

[4] K. P. Bogart, D. B. West, A short proof that proper=unit, *Discrete Mathematics* 201, (1999), 21-23.

[5] D. E. Brown, A. H. Busch, and G. Isaak, Linear time recognition algorithms and structure theorems for bipartite tolerance graphs and bipartite probe interval graphs, *Discrete Math. and Theoretical Comp. Sci.*, **12:5** (2010) 63-82.

[6] D. E. Brown, A. H. Busch, and J.R. Lundgren, Interval tournaments, *J Graph Theory* **56** (2007), 72-81.

[7] D. E. Brown, B. Flesch, J. R. Lundgren, Characterization of 2-tree probe interval graphs, Manuscripts, to appear.

[8] D. E. Brown, S. C. Flink, and J. R. Lundgren, Interval k-graphs, *Congressus Numerantium* **156** (2002), 5-16.

[9] D. E. Brown, S. C. Flink, and J. R. Lundgren, Characterizations of interval bigraphs and unit interval bigraphs. *Congressus Numerantium* **157** (2002), 79-93.

[10] D. E. Brown, L. Langley, Forbidden subgraph characterization of bipartite unit probe interval graphs, to appear.

[11] D. E. Brown, J. R. Lundgren, Bipartite probe interval graphs, circular arc graphs, and interval point bigraphs,*Australas J Combin* **35** (2006), 221-236.

[12] D. E. Brown, J. R. Lundgren, Several characterization of unit interval bigraphs, to appear (2011).

[13] D. E. Brown, J. R. Lundgren, Relationships Among Classes of Interval Bigraphs, (0,1)-Matrices, and Circular Arc Graphs.*Congressus Numerantium*, **166** (2004), 97-123.

[14] D. E. Brown, J. R. Lundgren, and C. Miller, Variations on interval graphs *Congressus Numerantium*, **149** (2001), 77-95.

[15] D. E. Brown, J. R. Lundgren and L. Sheng, A characterization of cycle-free unit probe interval graphs, *Discrete Applied Mathematics* **157**, (2009) 762-767.

[16] E. Drust, S. Dasgupta, J. R. Lundgren, A Class of interval digraphs, *Congressus Numerantium* **192** (2008), 97-113.

[17] D. R. Fulkerson and O.A. Gross, Incidence matrices and interval graphs, *Pacific J. Math. 15* (1965), 835855.

[18] F. Gardi, The Roberts characterization of proper and unit interval graphs. *Discrete Mathematics*, **307**, (2007) 2906-2908

[19] P. C. Gilmore and A. J. Hoffman, A characterization of comparability graphs and of interval graphs, *Canad. J. Math.* **16** (1964), 539 548.

[20] M. C. Golumbic, Algorithmic graph theory and perfect graphs, *annals of discrete mathenatics* **57**

[21] M.C. Golumbic and M. Lipshteyn, On the hierarchy of interval, probe, and tolerance graphs, *Congressus Numerantium* **153** (2001), 97106.

[22] G. Hajos, Uber eine Art von Graphen, *Internat. Math. Nachr*, **11** (1957), Problem 65.

[23] F. Harary, J. A. Kabell, and F. R. McMorris, *Comment. Math. Univ. Carolina*, **23(4)** (1982), 739-745.

[24] P. Hell and J. Huang, Interval Bigraphs and Circular Arc Graphs. Journal of Graph Theory 46 (2004) 313-327.

[25] L. Langley, J. R. Lundgren, Sarah K. Merz, The Competition Graphs of Interval Digraphs *Congr. Numer* **(1995**

[26] C. G. Lekkerkerker and J.C. Boland, Representations of finite graphs by a set of intervals on the real line, *Fund. Math.* **51** (1962), 45-64.

[27] I. Lin, M. Sen, D. West, Classes of interval digraphs and $0,1$-matrices. *Congressus Numberantium* **125** (1997), 201-209.

[28] J. R. Lundgren, B. Tonnsen, 2-trees that are interval k-graphs *Australas J Combin*, (2010).

[29] T. McKee, F. R. McMorris, Topics in intersection graph theory, *SIAM mongraphs on Dioscrete Mathematics and application*, *Society for Industrial and Applied Mathematics*, publ., Philadelphia, (1999).

[30] F. R. McMorris, C. Wang, and P. Zhang, On probe interval graphs, *Discrete Applied Mathematics* **88** (1998), 315324.

[31] H. Müller, Recognizing Interval Digraphs and Interval Bigraphs in Polynomial Time. Discrete Applied Mathematics 78 (1997) 189-205.

[32] A. Proskurowski, Separating subgraphs in k-trees, *Discrete Math.* **49** (1984) 275-285.

[33] A. Proskurowski and Jan Arne Telle, Classes of graphs with restricted interval model. Discrete Mathematics and Theoretical Com-puter Science, 3(1999), 167-176.

[34] N. Pržulj and D. G. Corneil, 2-Tree probe interval graphs have a large obstruction set, *Discrete Applied Mathematics* **150** (2005) 216-231.

[35] F. S. Roberts : Indifference graphs, In Harary, F., editor, *Proof Tech-niques in Graph Theory* pp. 139146. Academic Press, New York

[36] F. S. Roberts, Proof Techniques in Graph Theory. *Academic Press, New York, NY*, (1969) 139-146.

[37] F. S. Roberts, Discrete mathematical models, Prentice-Hall, Upper Saddle River, NJ, 1976.

[38] F. S. Roberts, Graph Theory and its applications to problems of Society, *CBMS NSF Regional conference series in Applied Mathematics.*

[39] M. Sen, S. Das, A. Roy, and D. West, Interval digraphs: An analogue of interval graphs, *J. Graph Theory* **13** (1989), 189-202.

[40] M. Sen, S. Das and D.B. West, Circular-arc digraphs: A characterization, *J Graph Theory* **13(4)** (1989), 581-592.

[41] L. Sheng, Cycle free probe interval graphs, *Congressus Numerantium* **140**, (1999), 33-42.

[42] D. B. West, Short proofs for interval digraphs,*Discrete Math* **178** (1998), 287-292.

[43] P. Zhang, Probe interval graph and its applications to physical mapping of DNA, Manuscript, 1994.

[44] P. Zhang, E.A. Schon, S.G. Fischer, E. Cayanis, J. Weiss, S. Kistler, and P.E. Bourne, An algorithm based on graph theory for the assembly of contigs in physical mapping of DNA, *CABIOS* **10** (1994), no. 3, 309317.

[45] Methods of mapping DNA fragments, United States Patent. http://www.cc.columbia.edu/cu/cie/techlists/patents/5667970.htm, 1997.